

The spectral flow for local boundary value problems on compact surfaces

Marina Prokhorova

Abstract

This paper deals with first order formally self-adjoint elliptic differential operators on a smooth compact oriented surface with non-empty boundary. We consider such operators with self-adjoint elliptic local boundary conditions. The paper is focused on paths in the space of such operators connecting two operators conjugated by a unitary automorphism. The first result of the paper is the computation of the spectral flow for such paths. The second result is the universality of the spectral flow on the space of loops of such operators: we show that every additive homotopy invariant vanishing on loops of invertible operators that takes values in a commutative group is a multiple of the spectral flow.

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1 Introduction

Local boundary value problems on a surface. This paper deals with first order formally self-adjoint elliptic differential operators on a smooth compact oriented surface M with non-empty boundary ∂M . We consider such operators with self-adjoint elliptic local boundary conditions (in particular, boundary conditions defined by general pseudo-differential operators are not allowed).

Let A be a first order formally self-adjoint elliptic differential operator acting on sections of a Hermitian vector bundle E over M . Denote by E_∂ the restriction of E to ∂M . A local boundary condition for A is defined by a smooth subbundle L of E_∂ ; the corresponding unbounded operator A_L on $L^2(M; E)$ has the domain

$$(1.1) \quad \text{dom}(A_L) = \{u \in H^1(M; E) : u|_{\partial M} \text{ is a section of } L\},$$

where $H^1(M; E)$ denotes the first order Sobolev space of sections of E .

The conormal symbol $\sigma(n)$ of A defines a symplectic structure on E_∂ . Since M is two-dimensional, E_∂ can be naturally decomposed into the direct sum $E_\partial^+ \oplus E_\partial^-$ of two Lagrangian subbundles. Namely, for every point $x \in \partial M$, the fibers E_x^+, E_x^- are the generalized eigenspaces of E_x corresponding to the eigenvalues of $\sigma(n)^{-1}\sigma(\xi)$ with positive and negative imaginary part respectively, where (n, ξ) is a positive oriented frame in T_x^*M .

A local boundary condition L is called elliptic for A if $L \cap E_\partial^+ = L \cap E_\partial^- = 0$; in this case A_L is a Fredholm operator. If, in addition, L is a Lagrangian subbundle of E_∂ , then A_L is self-adjoint. We denote by $\overline{\text{Ell}}(E)$ the space of all such pairs (A, L) . We equip $\overline{\text{Ell}}(E)$ with the C^1 -topology on symbols of operators, the C^0 -topology on their free terms, and the C^1 -topology on boundary conditions.

We show in Proposition 5.1 that self-adjoint elliptic local boundary conditions L for A are in a one-to-one correspondence with self-adjoint bundle automorphisms T of E_∂^- . This correspondence is given by the rule

$$(1.2) \quad L = \text{Ker } P_T \text{ with } P_T = P^+ (1 + i\sigma(n)^{-1}TP^-),$$

where P^+ denotes the projection of E_∂ onto E_∂^+ along E_∂^- and $P^- = 1 - P^+$. If A is the Dirac operator, then E_∂^+ and E_∂^- are mutually orthogonal; in this case L can be written as

$$(1.3) \quad L = \{u^+ \oplus u^- \in E_\partial^+ \oplus E_\partial^- : i\sigma(n)u^+ = Tu^-\}.$$

We associate with a pair $(A, L) \in \overline{\text{Ell}}(E)$ the subbundle $F = F(A, L)$ of E_∂^- , whose fibers F_x , $x \in \partial M$ are spanned by the generalized eigenspaces of T_x corresponding to negative eigenvalues.

The spectral flow. For a separable Hilbert space H , the space $\mathcal{R}^{\text{sa}}(H)$ of regular (that is, closed and densely defined) Fredholm self-adjoint operators on H equipped with the

gap topology is path-connected, and its fundamental group is isomorphic to \mathbb{Z} . This isomorphism is given by the 1-cocycle on $\mathcal{RF}^{\text{sa}}(H)$ called the spectral flow, which counts with signs the number of eigenvalues passing through zero from the start of the path to its end (the eigenvalues passing from negative values to positive one are counted with a plus sign, and the eigenvalues passing in the other direction are counted with a minus sign).

The map $(A, L) \mapsto A_L$ defines a natural inclusion $\overline{\text{Ell}}(E) \hookrightarrow \mathcal{RF}^{\text{sa}}(L^2(M; E))$. Hence the spectral flow is defined for paths in the space $\overline{\text{Ell}}(E)$ as well. Our first result is the computation of the spectral flow:

Theorem 1. *Let (A_t, L_t) be a path in $\overline{\text{Ell}}(E)$ such that (A_1, L_1) is conjugated with (A_0, L_0) by a unitary bundle automorphism g of E . Then the spectral flow of such a path is equal to the first Chern class of the vector bundle \mathcal{F} evaluated on the fundamental class of the product $\partial M \times S^1$:*

$$(1.4) \quad \text{sf}(A_t, L_t) = c_1(\mathcal{F})[\partial M \times S^1],$$

where \mathcal{F} is the vector bundle over $\partial M \times S^1$ obtained from the one-parameter family $F_t = F(A_t, L_t)$ of subbundles of E_∂ by gluing F_1 with F_0 by g .

This result was first announced by the author in [15, Section 8] (up to multiplication by an integer constant depending only on the homotopy type of M) and then in [16].

Universality of the spectral flow. The second result of the paper is the universality of the spectral flow for the spaces $\overline{\text{Ell}}(E)$. We prove it in two different settings: first for loops and then for paths with conjugate ends.

Denote by $\overline{\text{Ell}}^0(E)$ the subspace of $\overline{\text{Ell}}(E)$ consisting of all pairs (A, L) such that the unbounded operator A_L has no zero eigenvalues (since A_L is self-adjoint, this condition is equivalent to the invertibility of A_L). Recall that every complex vector bundle over M is trivial and that $\overline{\text{Ell}}(E)$ is empty for bundles E of odd rank. Denote by $2k_M$ the trivial vector bundle of rank $2k$ over M .

Theorem 2. *Let Λ be a commutative group. Suppose that we associate an element $\Phi(\gamma) \in \Lambda$ with every loop γ in $\overline{\text{Ell}}(2k_M)$ for every $k \in \mathbb{N}$. Then the following two conditions are equivalent:*

1. Φ is additive with respect to direct sums, homotopy invariant, and vanishing on loops in $\overline{\text{Ell}}^0(2k_M)$ for all k .
2. Φ has the form $\Phi(\gamma) = \text{sf}(\gamma) \cdot \lambda$ for some constant $\lambda \in \Lambda$.

Theorem 3 is the analogue of Theorem 2, but for paths (A_t, L_t) with conjugate ends instead of loops. The homotopy invariance in this case is understood to be the invariance with respect to the change of a path in the space $\Omega_g \overline{\text{Ell}}(E)$ of all paths in $\overline{\text{Ell}}(E)$ with ends conjugated by a *fixed* unitary automorphism g of E .

Theorem 3. *Let Λ be a commutative group. Suppose that we associate an element $\Phi(\gamma, g) \in \Lambda$ with every $\gamma \in \Omega_g \overline{\text{Ell}}(2k_M)$ for every $k \in \mathbb{N}$ and every smooth map $g: M \rightarrow \mathcal{U}(2k)$. Then the following two conditions are equivalent:*

1. Φ is additive with respect to direct sums, constant on path connected components of $\Omega_g \overline{\text{Ell}}(2k_M)$, and vanishing on $\Omega_g \overline{\text{Ell}}^0(2k_M)$ for all k, g .
2. Φ has the form $\Phi(\gamma, g) = \text{sf}(\gamma) \cdot \lambda$ for some constant $\lambda \in \Lambda$.

It is known that the spectral flow is a universal homotopy invariant for loops in the space $\mathcal{RF}^{\text{sa}}(H)$, and that the spectral flow is additive with respect to direct sums and vanishes on loops of invertible operators. But the space of first order self-adjoint elliptic local boundary problems for sections of E is a very small part of $\mathcal{RF}^{\text{sa}}(L^2(M; E))$. We cannot expect a priori from the spectral flow to be a universal invariant on this smaller space. Nevertheless, this is indeed the case if we consider vector bundles over M of various ranks altogether, as Theorems 2, 3 show.

Symbolic universality. As a byproduct of our proof of Theorems 1, 2, and 3, we obtain stronger, symbolic versions of Theorems 2 and 3, replacing the analytically defined subspace $\overline{\text{Ell}}^0(E)$ by smaller, symbolically defined subspaces.

Denote by $\text{Dir}(E)$ the subspace of $\text{Ell}(E)$ consisting of all Dirac operators which are odd with respect to chiral decomposition. Denote by $\overline{\text{Dir}}(E)$ the subspace of $\overline{\text{Ell}}(E)$ consisting of all pairs (D, L) such that $D \in \text{Dir}(E)$ and L is given by formula (1.3) with a unitary self-adjoint automorphism T . We consider three special subspaces of $\overline{\text{Dir}}(E)$. $\overline{\text{Dir}}_+(E)$ (resp. $\overline{\text{Dir}}_-(E)$) is the subspace of all $(D, L) \in \overline{\text{Dir}}(E)$ corresponding to $T = \text{Id}$ (resp. $T = -\text{Id}$). $\overline{\text{Dir}}_{\pm}(E)$ is the subspace of $\overline{\text{Dir}}(E)$ consisting of all operators (D, L) that can be written in the form $(D', L') \oplus (D'', L'')$ with $(D', L') \in \overline{\text{Dir}}_+(E')$ and $(D'', L'') \in \overline{\text{Dir}}_-(E'')$ for some orthogonal decomposition $E \cong E' \oplus E''$.

These three subspaces $\overline{\text{Dir}}_+(E)$, $\overline{\text{Dir}}_-(E)$, and $\overline{\text{Dir}}_{\pm}(E)$ of $\overline{\text{Ell}}(E)$ have purely symbolic descriptions, without reference to any analytic property of correspondent unbounded operators D_L . Nevertheless, these unbounded operators do share an important analytic property: all of them have no zero eigenvalues. (In fact, every D_L with $D \in \text{Dir}(E)$ and definite T has the same property, but we do not need all of them for our purposes.)

Theorem 4. *Let Λ be a commutative group. Suppose that we associate an element $\Phi(\gamma) \in \Lambda$ with every loop γ in $\overline{\text{Ell}}(2k_M)$ for every $k \in \mathbb{N}$. Then the following two conditions are equivalent:*

1. Φ is (a) additive with respect to direct sums, (b) homotopy invariant, and (c) vanishing on constant loops in $\overline{\text{Dir}}(2k_M)$ and on loops in $\overline{\text{Dir}}_{\pm}(2k_M)$ for every k .
2. Φ has the form $\Phi(\gamma) = \text{sf}(\gamma) \cdot \lambda$ for some constant $\lambda \in \Lambda$.

Theorem 5. *Let Λ be a commutative group. Suppose that we associate an element $\Phi(\gamma, g) \in \Lambda$*

with every $\gamma \in \Omega_g \overline{\text{Ell}}(2k_M)$ for every $k \in \mathbb{N}$ and every smooth map $g: M \rightarrow \mathcal{U}(2k)$. Then the following two conditions are equivalent:

1. Φ is (a) additive with respect to direct sums, (b) constant on path connected components of $\Omega_g \overline{\text{Ell}}(2k_M)$, and (c) vanishing on constant loops in $\overline{\text{Dir}}(2k_M)$, on $\Omega_g \overline{\text{Dir}}_+(2k_M)$, and on $\Omega_g \overline{\text{Dir}}_-(2k_M)$ for every k, g .
2. Φ has the form $\Phi(\gamma, g) = \text{sf}(\gamma) \cdot \lambda$ for some constant $\lambda \in \Lambda$.

Auxiliary theorems. The proofs of Theorems 1–5 are based on two auxiliary theorems.

Theorem 6. Suppose that Φ satisfies condition (1) of Theorem 4. Then Φ has the form

$$(1.5) \quad \Phi(\gamma) = c_1(\mathcal{F}(\gamma))[\partial M \times S^1] \cdot \lambda$$

for some constant $\lambda \in \Lambda$.

Theorem 7. Suppose that Φ satisfies condition (1) of Theorem 5. Then Φ has the form

$$(1.6) \quad \Phi(\gamma, g) = c_1(\mathcal{F}(\gamma, g))[\partial M \times S^1] \cdot \lambda$$

for some constant $\lambda \in \Lambda$.

The proofs of Theorems 6–7 are purely algebro-topological; we use no analytical means here.

To prove Theorem 6, we first show that (in the two-dimensional setting) there is a deformation retraction of $\overline{\text{Ell}}(E)$ onto a subspace of $\overline{\text{Dir}}(E)$. Then we show that if an additive homotopy invariant vanishes on loops in $\overline{\text{Dir}}_+(2k_M)$ for every k , then it factors through $\Omega \overline{\text{Ell}}(2k_M) \rightarrow K^0(\partial M \times S^1)$, $\gamma \mapsto [\mathcal{F}(\gamma)]$. Next we show that vanishing of Φ on constant loops cancels the image of $\pi^*: K^0(\partial M) \rightarrow K^0(\partial M \times S^1)$, while vanishing of Φ on loops in $\overline{\text{Dir}}_\pm(2k_M)$ cancels the image of $j^*: K^0(M \times S^1) \rightarrow K^0(\partial M \times S^1)$, where π denotes the projection of $\partial M \times S^1$ to ∂M and j denotes the embedding of $\partial M \times S^1$ to $M \times S^1$. But if a homomorphism from $K^0(\partial M \times S^1)$ to Λ vanishes on both $\text{Im } \pi^*$ and $\text{Im } j^*$, then it factors through the homomorphism $K^0(\partial M \times S^1) \rightarrow \mathbb{Z}$, $[\mathcal{F}] \mapsto c_1(\mathcal{F})[\partial M \times S^1]$. This implies (1.5).

The proof of Theorem 7 is going along the same lines. However, working with Ω_g instead of Ω gives us more freedom. This allows to weaken condition (1c) of Theorem 4, replacing the subspace $\overline{\text{Dir}}_\pm(2k_M)$ of $\overline{\text{Ell}}(2k_M)$ by the smaller subspace $\overline{\text{Dir}}_+(2k_M) \cup \overline{\text{Dir}}_-(2k_M)$.

Proofs of Theorems 1–5. To prove Theorem 1, we use the homotopy invariance of the spectral flow, its additivity with respect to direct sums, and the vanishing of the spectral flow on constant paths and on paths of invertible operators. The unbounded operator D_L is invertible for every $(D, L) \in \overline{\text{Dir}}_\pm(E)$, so the spectral flow vanishes on paths in $\overline{\text{Dir}}_\pm(E)$. Thus the spectral flow satisfies conditions of Theorem 7, with $\Phi = \text{sf}$ and

$\Lambda = \mathbb{Z}$. Theorem 7 implies that in the two-dimensional framework these properties alone are sufficient to compute the spectral flow (along paths with conjugate ends) up to an integer factor $\lambda = \lambda_M$. Simple reasoning shows that λ_M depends only on the diffeomorphism type of M . We then reduce the computation of λ_M to the case of the annulus. The factor λ_{ann} was computed by the author in [15] by direct evaluation. This gives $\lambda_M = \lambda_{\text{ann}} = 1$ for any surface M .

Theorems 2 and 4 are immediate corollaries of Theorems 1 and 6.

Theorems 3 and 5 are immediate corollaries of Theorems 1 and 7.

Gap continuity. In the Appendix we give some general criteria of being gap continuous for families of closed operators in Hilbert spaces. One particular case of these criteria, the case of boundary value problems for first order differential operators, is used in the main part of the paper, namely in the proofs of Proposition 4.1 and Lemma 8.4.

Motivation. The computation of the spectral flow for paths of first order self-adjoint elliptic operators over a surface is important for some applications in condensed matter physics. For example, the Aharonov-Bohm effect for a single-layer graphene sheet with holes arises if a one-parameter family of Dirac operators has non-zero spectral flow. The varying free term of the Dirac operator corresponds to a varying magnetic field, while the path connecting two gauge equivalent operators corresponds to the situation where magnetic fluxes through holes change by integer numbers in the units of the flux quantum. The spectral flow for such paths of Dirac operators was computed by the author in [15]. Later the results of [15] were improved and generalized to more general families of Dirac operators with local boundary conditions: for even-dimensional compact manifolds by A. Gorokhovsky and M. Lesch in [6], and for compact manifolds of arbitrary dimension by M. Katsnelson and V. Nazaiinskii in [11]. Unfortunately, the methods of both [6] and [11] use essentially the specific nature of Dirac operators and cannot be applied to general self-adjoint elliptic differential operators. However, some other possible realizations of the Aharonov-Bohm effect in condensed matter physics are described by self-adjoint elliptic operators of non-Dirac type. The initial motivation of the author in the present paper was to solve the arising mathematical problem, namely to compute the spectral flow for such a family.

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2 The spectral flow on the space of unbounded operators

The space of regular operators. Let H be a separable complex Hilbert space. Denote by $\mathcal{R}(H)$ the space of regular (= closed densely defined) operators on H with the gap topology. Recall that the gap topology on $\mathcal{R}(H)$ is induced by the gap metric $\delta(R_1, R_2) = \|P_1 - P_2\|$, where P_i denotes the orthogonal projection of $H \oplus H$ onto the graph of R_i . The subspace $\mathcal{B}(H)$ of $\mathcal{R}(H)$ consisting of the bounded operators is dense and open in $\mathcal{R}(H)$, and the induced topology on $\mathcal{B}(H)$ coincides with the usual norm topology [1, 4].

Denote by $\mathcal{RF}(H)$ the subspace of $\mathcal{R}(H)$ consisting of regular Fredholm operators, and by $\mathcal{RF}^{\text{sa}}(H)$ its subspace consisting of regular Fredholm self-adjoint operators. The space $\mathcal{RF}^{\text{sa}}(H)$ is path-connected and its fundamental group is isomorphic to \mathbb{Z} [8]. This isomorphism is given by the 1-cocycle on $\mathcal{RF}^{\text{sa}}(H)$ called the spectral flow. The definitions of the spectral flow can be found in [14] for the case of bounded operators and in [1, 13] for the case of unbounded operators.

The case where one or both of the endpoints of the path have zero eigenvalue requires some agreement on the counting procedure. Yet if a path is a loop up to an automorphism of H , the value of the spectral flow is independent of the choice of definition. Since we consider only such paths in this paper, we do not specify the counting agreement for the case of non-invertible endpoints: any such agreement will suffice.

The properties of the spectral flow. It is well known that the spectral flow has a number of nice properties:

(So) Zero crossing. In the absence of zero crossing the spectral flow vanishes: if γ is a continuous path in $\mathcal{RF}^{\text{sa}}(H)$ such that 0 is not an eigenvalue of $\gamma(t)$ for any t , then $\text{sf}(\gamma) = 0$.

(So') The spectral flow of a constant path vanishes.

(S1) Additivity with respect to direct sum. Let H_0, H_1 be separable Hilbert spaces, and let $\gamma_i: [a, b] \rightarrow \mathcal{RF}^{\text{sa}}(H_i)$ be continuous paths. Then $\text{sf}(\gamma_0 \oplus \gamma_1) = \text{sf}(\gamma_0) + \text{sf}(\gamma_1)$, where $\gamma_0 \oplus \gamma_1: [a, b] \rightarrow \mathcal{RF}^{\text{sa}}(H_0 \oplus H_1)$ denotes the pointwise direct sum of paths.

(S2) Homotopy invariance. The spectral flow along the continuous path γ in $\mathcal{RF}^{\text{sa}}(H)$ does not change if γ changes continuously in the space of paths in $\mathcal{RF}^{\text{sa}}(H)$ with fixed endpoints (the same as the endpoints of γ).

(S3) Path additivity. Suppose γ, γ' are continuous paths in $\mathcal{RF}^{\text{sa}}(H)$ such that the last point of γ is the first point of γ' . Then $\text{sf}(\gamma \cdot \gamma') = \text{sf}(\gamma) + \text{sf}(\gamma')$, where $\gamma \cdot \gamma'$ denotes the concatenation of γ and γ' .

(S4) Conjugacy invariance. Let g be a unitary automorphism of H , and let γ be a continuous path in $\mathcal{RF}^{\text{sa}}(H)$. Then $\text{sf}(\gamma) = \text{sf}(g\gamma g^{-1})$.

In this paper we compute the spectral flow only for paths with conjugate ends (in particular, for loops), so it is convenient to have a special designation for the space of such paths. For a topological space X we denote by ΩX the space of free loops with the compact-open topology. Here by a free loop we mean a continuous map from a circle S^1 to X , or, equivalently, a continuous map $\gamma: [0, 1] \rightarrow X$ such that $\gamma(0) = \gamma(1)$. For a homeomorphism g of X we denote by $\Omega_g X$ the space of continuous paths $\gamma: [0, 1] \rightarrow X$ such that $\gamma(1) = g\gamma(0)$, with the compact-open topology. We say that $\gamma, \gamma' \in \Omega_g X$ are homotopic if they can be connected by a path in $\Omega_g X$.

The group $U(H)$ of unitary automorphisms of H acts on the space $\mathcal{RF}^{\text{sa}}(H)$ by conjugations: $(A, g) \mapsto gAg^{-1}$. We will write $\Omega_g \mathcal{RF}^{\text{sa}}(H)$ for $g \in U(H)$ having in mind this action.

In the proof of Theorem 1 we do not use all properties (So-S4) but only the following small part of them:

(So^U) Zero crossing. Let g be a unitary automorphism of H , and let $\gamma \in \Omega_g \mathcal{RF}^{\text{sa}}(H)$. Suppose that $\gamma(t)$ has no zero eigenvalue for each $t \in [0, 1]$. Then $\text{sf}(\gamma) = 0$.

(S1^U) Additivity with respect to direct sum. Let g_i be a unitary automorphism of H_i , and let $\gamma_i \in \Omega_{g_i} \mathcal{RF}^{\text{sa}}(H_i)$, $i = 0, 1$. Then $\text{sf}(\gamma_0 \oplus \gamma_1) = \text{sf}(\gamma_0) + \text{sf}(\gamma_1)$.

(S2^U) Homotopy invariance. Let g be a unitary automorphisms of H . If $\gamma_0, \gamma_1 \in \Omega_g \mathcal{RF}^{\text{sa}}(H)$ are connected by a path in $\Omega_g \mathcal{RF}^{\text{sa}}(H)$, then $\text{sf}(\gamma_0) = \text{sf}(\gamma_1)$.

Proof. Properties (So^U) and (S1^U) are just weaker versions of (So) and (S1) respectively. To prove (S2^U), we combine (S2), (S3), (S4), and (So'). Let $\gamma_s(t)$, $s \in [0, 1]$ be a homotopy between γ_0 and γ_1 . Let the paths $\beta, \beta', \beta'': [0, 1] \rightarrow \mathcal{RF}^{\text{sa}}(H)$ be given by the formulas $\beta(t) = \gamma_t(0)$, $\beta'(t) = \gamma_1(t)$, $\beta''(t) = \gamma_{1-t}(1)$. Then γ_0 is homotopic to $\beta \cdot \beta' \cdot \beta''$ in the space of paths in $\mathcal{RF}^{\text{sa}}(H)$ with the same endpoints as γ_0 . Property (S2) implies $\text{sf}(\gamma_0) = \text{sf}(\beta \cdot \beta' \cdot \beta'')$, and by (S3) the last value is equal to $\text{sf}(\beta) + \text{sf}(\beta') + \text{sf}(\beta'')$. Property (S4) implies $\text{sf}(\beta) = \text{sf}(g\beta)$. The path $g\beta$ is just the path β'' passing in the opposite direction, so the concatenation of these two paths is homotopic to the constant path (in the class of paths with fixed endpoints). By (S3), (S2), and (So') we have $\text{sf}(g\beta) + \text{sf}(\beta') = 0$. Taking all this together, we obtain $\text{sf}(\gamma_0) = \text{sf}(\beta') = \text{sf}(\gamma_1)$. \square

3 Elliptic local boundary value problem

Throughout the paper (except for the Appendix) M is a smooth compact connected oriented surface with non-empty boundary ∂M and a fixed Riemannian metric.

Operators. Let A be a first order formally self-adjoint elliptic differential operator acting on sections of a smooth Hermitian complex vector bundle E over M . Recall that an operator A is called elliptic if its (principal) symbol $\sigma_A(\xi)$ is non-degenerate for every

non-zero cotangent vector $\xi \in T^*M$. An operator A is called formally self-adjoint if it is symmetric on the domain $C_0^\infty(M; E)$, that is, if $\int_M \langle Au, v \rangle ds = \int_M \langle u, Av \rangle ds$ for any smooth sections u, v of E with compact supports in $M \setminus \partial M$. Throughout the paper (except for the Appendix) all differential operators are supposed to have smooth (C^∞) coefficients.

Local boundary conditions. The differential operator A defines a linear operator on the subspace $C_0^\infty(M; E)$ of the Hilbert space $L^2(M; E)$. This linear operator can be extended to an (unbounded) regular self-adjoint operator on $L^2(M; E)$ by imposing appropriate boundary conditions. We will consider only local boundary conditions. Denote by E_∂ the restriction of E to the boundary ∂M of M . A smooth subbundle L of E_∂ defines a local boundary condition for A ; the corresponding unbounded operator A_L on $L^2(M; E)$ has the domain

$$(3.1) \quad \text{dom}(A_L) = \{u \in H^1(M; E) : u|_{\partial M} \text{ is a section of } L\},$$

where $H^1(M; E)$ denotes the first order Sobolev space (the space of sections of E which are in L^2 together with all their first derivatives). We will often identify a pair (A, L) with the operator A_L .

Decomposition of E . To describe when a subbundle L is an “appropriate boundary condition”, give first some properties of self-adjoint elliptic symbols.

Proposition 3.1. *Let A be a first order formally self-adjoint elliptic differential operators acting on sections of E , and let σ be the symbol of A . Then for any $x \in M$ and any positive oriented frame (e_1, e_2) in T_x^*M the operator*

$$Q_x = \sigma(e_1)^{-1} \sigma(e_2) \in \text{End}(E_x)$$

*has no eigenvalues on the real axis. Let E_x^+ and E_x^- denote the generalized eigenspaces of E_x corresponding to the eigenvalues of Q_x with positive and negative imaginary part respectively. These eigenspaces are independent of the choice of a positive oriented frame (e_1, e_2) and form the fibers of smooth subbundles $E^+ = E^+(\sigma)$ and $E^- = E^-(\sigma)$ of E , so we have the direct sum decomposition $E = E^+ \oplus E^-$ (not necessarily orthogonal). Ranks of E^+ and E^- are equal, so the rank of E is even. Finally, $\sigma(\xi)E_x^\pm = (E_x^\pm)^\perp$ for any non-zero $\xi \in T_x^*M$.*

Proof. Since A is elliptic, the operator $Q_x - t = \sigma(e_1)^{-1} \sigma(e_2 - te_1)$ is invertible for any $t \in \mathbb{R}$. Hence Q_x has no eigenvalues on the real axis and $E_x = E_x^+ \oplus E_x^-$.

If we change (e_1, e_2) to $(e_1, e_2 + te_1)$, $t \in \mathbb{R}$, then Q_x is changed to $Q_x + t \text{Id}$. If we change (e_1, e_2) to $(e_1 + te_2, e_2)$ then Q_x is changed to $(Q_x^{-1} + t \text{Id})^{-1}$. In both cases E_x^+, E_x^- do not change. Therefore, they do not change at any change of the frame (e_1, e_2) preserving orientation, and thus depend only on x . Moreover, E_x^+ and E_x^- smoothly depend on Q_x and therefore on x , so they are fibers of smooth vector bundles E^+ and E^- .

Let $\xi \in T_x^*M$ be a non-zero cotangent vector. Choose a positive oriented frame (e_1, e_2) in T_x^*M such that $e_1 = \xi$. Denote $\sigma_i = \sigma(e_i)$, $V_{\lambda, k} = \text{Ker}(Q_x - \lambda)^k$, and $V_\lambda = V_{\lambda, \dim E_x}$. We prove by induction that $\sigma_1 V_\lambda$ is orthogonal to V_μ for any $\lambda, \mu \in \mathbb{C}$ with $\lambda \neq \bar{\mu}$. Indeed, $\sigma_1 V_{\lambda, 0} = 0$ is orthogonal to $V_{\mu, 0} = 0$. Suppose that $\sigma_1 V_{\lambda, l}$ is orthogonal to $V_{\mu, m}$ for all $l, m \geq 0$, $l + m < k$. Then for $l + m = k$, $u \in V_{\lambda, l}$, $v \in V_{\mu, m}$ we have

$$\begin{aligned} (\lambda - \bar{\mu}) \langle \sigma_1 u, v \rangle &= \langle \sigma_1 \lambda u, v \rangle - \langle \sigma_1 u, \bar{\mu} v \rangle + \langle u, \sigma_2 v \rangle - \langle \sigma_2 u, v \rangle = \\ &= \langle \sigma_1 \lambda u, v \rangle - \langle \sigma_1 u, \bar{\mu} v \rangle + \langle \sigma_1 u, Qv \rangle - \langle \sigma_1 Qu, v \rangle = \langle \sigma_1 u, (Q - \bar{\mu})v \rangle - \langle \sigma_1 (Q - \lambda)u, v \rangle = 0 \end{aligned}$$

by induction assumption, since $(Q - \bar{\mu})v \in V_{\mu, m-1}$ and $(Q - \lambda)u \in V_{\lambda, l-1}$. Thus $\sigma_1 V_\lambda$ is orthogonal to V_μ if $\lambda \neq \bar{\mu}$.

E_x^+ is spanned by $\bigcup V_\lambda$ with λ running all eigenvalues of Q_x with positive imaginary parts. For every pair λ, μ of such eigenvalues (not necessarily distinct) we have $\lambda \neq \bar{\mu}$, so $\sigma_1 E_x^+$ is orthogonal to E_x^+ . Similarly, $\sigma_1 E_x^-$ is orthogonal to E_x^- . We have

$$2 \dim E_x^+ = \dim E_x^+ + \dim(\sigma_1 E_x^+) \leq \dim E_x^+ + \dim(E_x^+)^{\perp} = \dim E_x$$

and, similarly, $2 \dim E_x^- \leq \dim E_x$. On the other hand, $\dim E_x^+ + \dim E_x^- = \dim E_x$. Therefore, $\dim E_x^+ = \dim E_x^- = \dim E_x/2$. This completes the proof of the proposition. \square

Self-adjoint elliptic boundary conditions. Now we are ready to describe appropriate boundary conditions for A . Denote $E_\partial^- = E^-|_{\partial M}$, $E_\partial^+ = E^+|_{\partial M}$. The conormal symbol $\sigma(n)$ of A defines a symplectic structure on the fibers of E_∂ given by the symplectic 2-form $\omega_x(u, v) = \langle i\sigma(n)u, v \rangle$ for $u, v \in E_x$, $x \in \partial M$, where n is the outward conormal to ∂M . With respect to this symplectic structure, E_∂^+ and E_∂^- are Lagrangian subbundles of E_∂ .

A smooth subbundle L of E_∂ is an **elliptic boundary condition** for A (or, what is one and the same, Shapiro-Lopatinskii boundary condition) if

$$(3.2) \quad L \cap E_\partial^+ = L \cap E_\partial^- = 0.$$

If additionally L is a Lagrangian subbundle of E_∂ , that is

$$(3.3) \quad \sigma(n)L = L^{\perp},$$

then L is a **self-adjoint boundary condition** for A .

Proposition 3.2. *Let A be a first order formally self-adjoint elliptic differential operators acting on sections of E , and let L be a smooth subbundle of E_∂ satisfying conditions (3.2), (3.3). Then A_L is a regular Fredholm self-adjoint operator on $L^2(M; E)$.*

Proof. Denote by τ the trace map from $H^1(M; E)$ to $H^{1/2}(\partial M; E_\partial)$. Let P be the bundle endomorphism projecting E_∂ on L^{\perp} along L . The operator $A \oplus P\tau: H^1(M; E) \rightarrow L^2(M; E) \oplus$

$H^{1/2}(\partial M; L^\perp)$ is Fredholm by [7, Theorem 20.1.2]. The operator $A_L: \text{dom}(A_L) = \ker(P\tau) \rightarrow L^2(M; E)$ coincides with the restriction of A to the kernel of $P\tau$, so A_L is also Fredholm. Every Fredholm operator is closed. The domain (3.1) is dense in $L^2(M; E)$. Green's formula for A implies that A_L is symmetric. It remains to show self-adjointness of A_L . Let u be an arbitrary element of the domain of the adjoint operator. This means that for each $w \in \text{dom}(A_L)$ we have $\langle u, Aw \rangle = \langle v, w \rangle$, where $v = (A_L)^*u \in L^2(M; E)$. By [9, Theorem 1], (u, v) lies in the closure of the graph of A_L (the statement of this theorem concerns only smooth domains in Euclidean spaces, but the same proof holds for arbitrary smooth manifolds without change.) Since A_L is closed, $u \in \text{dom}(A_L)$. Therefore, A_L is self-adjoint. This completes the proof of the proposition. \square

Remark. If the dimension of M is greater than 2, then condition (3.2) should be replaced by the following condition: $L \cap E^-(\xi) = 0$ for each $\xi \in T_x^* \partial M \setminus \{0\}$, $x \in \partial M$. Here $E^-(\xi)$ denotes the generalized eigenspace of E_x corresponding to the eigenvalues of $\sigma(n_x)^{-1} \sigma(\xi)$ with negative imaginary part. In the 2-dimensional case, however, $T_x^* \partial M \setminus \{0\}$ consist of only two rays, so the identity $E^-(-\xi) = E^+(\xi)$ allows to write the ellipticity condition in form (3.2).

4 The space of elliptic operators

Denote by $\text{Ell}(E)$ the set of first order formally self-adjoint elliptic differential operators acting on sections of E . Recall that all operators are supposed to have smooth coefficients. Denote by $\overline{\text{Ell}}(E)$ the set of all pairs (A, L) such that $A \in \text{Ell}(E)$ and L is a smooth subbundle of E_∂ satisfying conditions (3.2), (3.3).

For a complex vector bundle V , we denote by $\text{Gr}(N; V)$ the smooth bundle over N whose fiber over $x \in N$ is the complex Grassmanian $\text{Gr}(V_x)$. In the same manner we define the smooth bundle $\text{End}(V)$ of fiber endomorphisms.

Let $r = (r_1, r_0)$ be a couple of integers, $r_1 \geq r_0 \geq 0$. Denote by $\text{Ell}^r(E)$ the set $\text{Ell}(E)$ equipped with the C^{r_1} -topology on symbols and the C^{r_0} -topology on free terms of operators. To be more precise, notice that the tangent bundle TM is trivial since M is a surface with non-empty boundary. Therefore we can choose smooth global sections e_1, e_2 of TM such that $e_1(x), e_2(x)$ are linear independent for any $x \in M$. Choose a smooth unitary connection ∇ on E . Each $A \in \text{Ell}(E)$ can be written uniquely as $A = \sigma_1 \nabla_1 + \sigma_2 \nabla_2 + a$, where the symbol components $\sigma_i = \sigma_A(e_i)$ are self-adjoint automorphisms of E , $\nabla_i = \nabla_{e_i}$, and the free term a is a bundle endomorphism. Therefore the choice of (e_1, e_2, ∇) defines the inclusion

$$\text{Ell}(E) \hookrightarrow C^{r_1}(M; \text{End}(E))^2 \times C^{r_0}(M; \text{End}(E)), \quad \sigma_1 \nabla_1 + \sigma_2 \nabla_2 + a \mapsto (\sigma_1, \sigma_2, a).$$

Equip $\text{Ell}(E)$ with the topology induced by this inclusion and denote the resulting space by $\text{Ell}^r(E)$. Equip $\overline{\text{Ell}}(E)$ with the topology induced by the inclusion $\overline{\text{Ell}}(E) \hookrightarrow \text{Ell}^r(E) \times$

$C^1(\partial M; \text{Gr}(E_\partial))$ (with the product topology on the last space) and denote the resulting space by $\overline{\text{Ell}}^r(E)$. Thus defined topologies on $\text{Ell}^r(E)$, $\overline{\text{Ell}}^r(E)$ are independent of the choice of a frame (e_1, e_2) and connection ∇ .

Proposition 4.1. *The natural inclusion $\overline{\text{Ell}}^{(0,0)}(E) \hookrightarrow \mathcal{R}\mathcal{F}^{\text{sa}}(L^2(M; E))$, $(A, L) \mapsto A_L$ is gap continuous.*

This proposition is proved in the end of the Appendix. Obviously, it implies that the inclusion $\overline{\text{Ell}}^r(E) \hookrightarrow \mathcal{R}\mathcal{F}^{\text{sa}}(L^2(M; E))$ is continuous for each $r_1 \geq r_0 \geq 0$.

Convention. *From now on we will use the $(1, 0)$ -topology on $\overline{\text{Ell}}(E)$, that is, the C^1 -topology on symbols and the C^0 -topology on free terms of operators. For brevity we will omit the superscript, so further $\overline{\text{Ell}}(E)$ will always mean $\overline{\text{Ell}}^{(1,0)}(E)$.*

Remark. We chose to use the stronger $(1, 0)$ -topology on $\overline{\text{Ell}}(E)$ instead of the $(0, 0)$ -topology to simplify the proofs. Probably, all theorems in the paper remain valid for $(0, 0)$ -topology on $\overline{\text{Ell}}(E)$ as well, but the author did not check this. It can be easily seen that Theorems 2, 3, 4, and 5 are valid (and their proofs remain the same) for (r_1, r_0) -topology on $\overline{\text{Ell}}(E)$ with $r_1 - 1 \geq r_0 \geq 0$.

5 The spectral flow for elliptic operators on a surface

Let γ be a continuous path in $\overline{\text{Ell}}(E)$. By Propositions 3.2 and 4.1, the correspondent path of regular operators is a continuous path in $\mathcal{R}\mathcal{F}^{\text{sa}}(L^2(M; E))$, so the spectral flow $\text{sf}(\gamma)$ along this path is well defined.

Denote by $\text{U}(E)$ the group of smooth unitary bundle automorphisms of E . We do not use any topology on $\text{U}(E)$, so we consider it as a discrete group.

Each $g \in \text{U}(E)$ defines a unitary automorphism of $L^2(M; E)$. The correspondent action of $\text{U}(E)$ on $\overline{\text{Ell}}(E)$ is given by the rule $g(A, L) = (gAg^{-1}, gL)$. Obviously, the spectrum of $g(A, L)$ coincides with the spectrum of (A, L) .

Let $\gamma: [0, 1] \rightarrow \overline{\text{Ell}}(E)$, $\gamma = (A_t, L_t)$ be a continuous path such that $\gamma(1) = g\gamma(0)$. The first main purpose of the paper is a computation of the spectral flow along such path. We give an answer in terms of the Chern number of the vector bundle \mathcal{F} over $\partial M \times S^1$, which we construct by a canonical way from γ and g . Namely, with every $(A, L) \in \overline{\text{Ell}}(E)$ we associate the vector subbundle $F(A, L)$ of E_∂ . The family (F_t) , $F_t = F(A_t, L_t)$ of subbundles of E_∂ defines the vector bundle $\widehat{F} = \widehat{F}(\gamma)$ over $\partial M \times [0, 1]$. Gluing F_1 with F_0 by g , we obtain the vector bundle $\mathcal{F} = \mathcal{F}(\gamma, g)$ over $\partial M \times S^1$.

The correspondence between subbundles of E_∂ and automorphisms of E_∂^- . To define

$F(A, L)$, we need some preparations. Let $A \in \text{Ell}(E)$, $E^+ = E^+(A)$, $E^- = E^-(A)$, and E_∂^+ , E_∂^- are the restrictions of E^+ , E^- to the boundary.

Suppose for a moment that E_∂^+ and E_∂^- are *mutually orthogonal* subbundles of E_∂ (this holds, in particular, for the Dirac operators). With every subbundle $L \subset E_\partial$ of half rank transversal to both E_∂^+ and E_∂^- we can associate the projection of E_∂^- onto E_∂^+ along L . Composing this projection with $-i\sigma(n): E_\partial^+ \rightarrow (E_\partial^+)^\perp = E_\partial^-$, we obtain the bundle automorphism T of E_∂^- . Conversely, with every bundle automorphism T of E_∂^- we associate the subbundle L of E_∂ given by the formula

$$(5.1) \quad L = \{u^+ \oplus u^- \in E_\partial^+ \oplus E_\partial^- = E_\partial: i\sigma(n)u^+ = Tu^-\}.$$

The automorphism T is self-adjoint if and only if L is Lagrangian, so we obtain a bijection between the set of all self-adjoint elliptic local boundary conditions for A and the set of all self-adjoint bundle automorphisms of E_∂^- .

This simple trick does not work in the general case, where E_∂^+ and E_∂^- are not mutually orthogonal. However, we can modify it to obtain such a bijection for the general case as well, though with a bit more complicated definition, as the following proposition shows.

Denote by P^+ the projection of E_∂ onto E_∂^+ along E_∂^- and by $P^- = 1 - P^+$ the projection of E_∂ onto E_∂^- along E_∂^+ .

Proposition 5.1.

1. *There is a one-to-one correspondence between smooth automorphisms T of E_∂^- and smooth subbundles L of E_∂ of half rank satisfying condition $L \cap E_\partial^+ = L \cap E_\partial^- = 0$. This correspondence is given by the formula*

$$(5.2) \quad L = \text{Ker } P_T \text{ with } P_T = P^+ (1 + i\sigma(n)^{-1}TP^-);$$

here P_T is the projection of E_∂ onto E_∂^+ along L .

2. *For L and T as above, L is Lagrangian if and only if T is self-adjoint.*

If E_∂^+ and E_∂^- are mutually orthogonal, then (5.2) is equivalent to (5.1).

In the rest of the paper we will sometimes write an element of $\overline{\text{Ell}}(E)$ as (A, T) instead of (A, L) .

Proof. Let us consider the following diagram:

$$(5.3) \quad \begin{array}{ccccc} L & \xrightarrow{P^+} & E_\partial^+ & \xrightarrow{i\sigma(n)} & (E_\partial^+)^\perp \\ P^- \downarrow & & & & \uparrow (P^-)^* \\ E_\partial^- & \xrightarrow{\quad T \quad} & E_\partial^- & & \end{array}$$

Here $(P^-)^*$ is the projection of E_∂ onto $(E_\partial^+)^\perp$ along $(E_\partial^-)^\perp$.

1. Let L be a smooth subbundle of E_∂ of half rank such that $L \cap E_\partial^+ = L \cap E_\partial^- = 0$. Then all solid arrows in the diagram are smooth bundle isomorphisms, so there is a smooth automorphism T of E_∂^- making this diagram commutative, and such automorphism is unique. For $u \in E_\partial$, we have $u \in L$ if and only if

$$(5.4) \quad (i\sigma(n)P^+ - TP^-)u \in (E_\partial^-)^\perp.$$

Multiplying by $-i\sigma(n)^{-1}$, we obtain the equivalent condition

$$(P^+ + i\sigma(n)^{-1}TP^-)u \in E_\partial^-.$$

Since $E_\partial^- = \text{Ker } P^+$, this is equivalent to $P_T u = 0$. Thus $L = \text{Ker } P_T$.

Conversely, let T be a smooth automorphism of E_∂^- . Obviously, $\text{Im } P_T \subset E_\partial^+$, the restriction of P_T to E_∂^+ is an identity, and $P_T^2 = P_T$. Therefore, P_T is the projection of E_∂ onto E_∂^+ along $L = \text{Ker } P_T$. The projection P_T smoothly depends on $x \in \partial M$, so L is a smooth subbundle of E_∂ with $\text{rank } L = \text{rank } E_\partial - \text{rank } E_\partial^+ = \text{rank } E_\partial/2$. If $u \in L \cap E_\partial^+$, then $u = P_T u = 0$, so $L \cap E_\partial^+ = 0$. If $u \in L \cap E_\partial^-$, then $i\sigma(n)^{-1}Tu \in E_\partial^-$ and $u = 0$, so $L \cap E_\partial^- = 0$. This completes the proof of clause 1.

2. Let L, T be as in clause 1. For any $u^-, v^- \in E_\partial^-$, $x \in \partial M$, there are $u^+, v^+ \in E_\partial^+$ such that both $u^- + u^+, v^- + v^+$ lie in L_x . Since E_∂^+ and E_∂^- are Lagrangian, from (5.4) we have

$$(5.5) \quad \begin{aligned} \langle Tu^-, v^- \rangle - \langle u^-, Tv^- \rangle &= \langle i\sigma(n)u^+, v^- \rangle - \langle u^-, i\sigma(n)v^+ \rangle = \\ &= \langle i\sigma(n)u^+, v^- \rangle + \langle i\sigma(n)u^-, v^+ \rangle = \langle i\sigma(n)(u^- + u^+), v^- + v^+ \rangle. \end{aligned}$$

If L is Lagrangian, then the last scalar product vanishes, so T is self-adjoint.

Conversely, suppose that T is self-adjoint. Repeating computation (5.5) in reverse order, we have

$$\forall u, v \in L_x \quad \langle i\sigma(n)u, v \rangle = \langle TP^-u, P^-v \rangle - \langle P^-u, TP^-v \rangle = 0,$$

so L_x is isotropic. But $\text{rank } L = \text{rank } E_\partial/2$, so L is Lagrangian. This completes the proof of the proposition. \square

The definition of $F(A, L)$. The map F from the space $\overline{\text{Ell}}(E)$ to the space of smooth subbundles of E_∂ is defined as follows. Let $(A, L) \in \overline{\text{Ell}}(E)$. First, take the subbundles $E_\partial^+(A), E_\partial^-(A)$ of E_∂ as described in Proposition 3.1. Second, define the self-adjoint automorphism T of E_∂^- by formula (5.2). Finally, define F_x as the invariant subspace of T_x spanned by the generalized eigenspaces of T_x corresponding to negative eigenvalues. Subspaces F_x of E_∂^- smoothly depend on $x \in \partial M$ and therefore are fibers of the smooth subbundle $F = F(A, L)$ of E_∂^- . Being a subbundle of E_∂^- , $F(A, L)$ is also a smooth subbundle of E_∂ .

In the next section we prove that the map $F: \overline{\text{Ell}}(E) \rightarrow C^1(\partial M; \text{Gr}(E_\partial))$ is continuous (see Lemma 7.1).

The definition of $\mathcal{F}(\gamma, g)$. Let $\gamma: [0, 1] \rightarrow \overline{\text{Ell}}(E)$ be a continuous path such that $\gamma(1) = g\gamma(0)$ for some $g \in \text{U}(E)$. With every such pair (γ, g) we associate the subbundle $\mathcal{F}(\gamma, g)$ of \mathcal{E}_∂ by the following rule. First, lift E_∂ to the vector bundle $\widehat{E}_\partial = E_\partial \times [0, 1] \rightarrow \partial M \times [0, 1]$. Then form the vector bundle \mathcal{E}_∂ over $\partial M \times S^1$ as the factor of \widehat{E}_∂ , identifying $(u, 1)$ with $(gu, 0)$ for every $u \in E_\partial$. The continuous family $F_t = F(\gamma(t))$ of subbundles of E_∂ forms the subbundle \widehat{F} of \widehat{E}_∂ . The condition $\gamma(1) = g\gamma(0)$ implies $F_1 = gF_0$, so \widehat{F} descends onto $\partial M \times S^1$ giving the subbundle $\mathcal{F} = \mathcal{F}(\gamma, g)$ of \mathcal{E}_∂ such that the following diagram is commutative:

$$\begin{array}{ccccc} \widehat{F} & \hookrightarrow & \widehat{E}_\partial & \longrightarrow & \partial M \times [0, 1] \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{F} & \hookrightarrow & \mathcal{E}_\partial & \longrightarrow & \partial M \times S^1 \end{array}$$

If $g = \text{Id}$, then we will write $\mathcal{F}(\gamma)$ instead of $\mathcal{F}(\gamma, \text{Id})$.

Theorem 1. Let $\gamma: [0, 1] \rightarrow \overline{\text{Ell}}(E)$ be a continuous path such that $\gamma(1) = g\gamma(0)$ for some smooth unitary bundle automorphism g of E . Then

$$(5.6) \quad \text{sf}(\gamma) = c_1(\mathcal{F}(\gamma, g))[\partial M \times S^1].$$

Here $c_1(\mathcal{F})$ is the first Chern class of the vector bundle \mathcal{F} , $[\partial M \times S^1]$ is the fundamental class of $\partial M \times S^1$, and ∂M is equipped with an orientation in such a way that the pair (outward normal to ∂M , positive tangent vector to ∂M) has a positive orientation.

Point out that we *do not* require weak inner unique continuation property for operators (A_t, L_t) . While Dirac operators always have this property, for general first order self-adjoint elliptic operators this is not necessarily so.

We postpone the proof of this theorem to Section 8.

6 Universality of the spectral flow

Denote by $\overline{\text{Ell}}^0(E)$ the subspace of $\overline{\text{Ell}}(E)$ consisting of all pairs (A, L) such that the unbounded operator A_L has no zero eigenvalues.

Recall that every Hermitian vector bundle E over M is trivial and that $\text{Ell}(E)$ is empty for bundles E of odd rank. Denote by $\overline{\text{Ell}}_M$ the disjoint union of $\overline{\text{Ell}}(2k_M)$ for all $k \in \mathbb{N}$, where $2k_M$ denotes the trivial vector bundle $\mathbb{C}^{2k} \times M \rightarrow M$ with a standard Hermitian structure on \mathbb{C}^{2k} . The space $\overline{\text{Ell}}_M$ has the natural structure of a graded topological

monoid, with the monoid operation given by the direct sum of operators and boundary conditions. Denote by $\overline{\text{Ell}}_M^0$ the disjoint union of $\overline{\text{Ell}}^0(2k_M)$ for all $k \in \mathbb{N}$; it is a submonoid of $\overline{\text{Ell}}_M$.

Loops. Recall that by $\Omega \overline{\text{Ell}}_M$ we denoted the space of continuous maps $\gamma: S^1 \rightarrow \overline{\text{Ell}}_M$ with the compact-open topology. The loop space $\Omega \overline{\text{Ell}}_M = \coprod_{k \geq 0} \Omega \overline{\text{Ell}}(2k_M)$ has the natural structure of a graded topological monoid, with the monoid operation given by the pointwise direct sum: $\gamma \oplus \gamma'$ is the loop in $\overline{\text{Ell}}_M$ given by the formula $t \mapsto \gamma(t) \oplus \gamma'(t)$ for $\gamma, \gamma' \in \Omega \overline{\text{Ell}}_M$.

The spectral flow defines the map $\text{sf}: \Omega \overline{\text{Ell}}_M \rightarrow \mathbb{Z}$ which is homotopy invariant, additive with respect to direct sums, and vanishing on loops of operators without zero eigenvalues.

Let us consider more general situation. Suppose that we have a function Φ which maps $\Omega \overline{\text{Ell}}_M$ to some commutative group Λ . We are interested in functions Φ satisfying the following three properties:

(Po) Zero crossing. $\Phi(\gamma) = 0$ for every $\gamma \in \Omega \overline{\text{Ell}}_M^0$.

(P1) Additivity with respect to direct sum. $\Phi(\gamma \oplus \gamma') = \Phi(\gamma) + \Phi(\gamma')$ for every $\gamma, \gamma' \in \Omega \overline{\text{Ell}}_M$.

(P2) Homotopy invariance. $\Phi(\gamma) = \Phi(\gamma')$ if γ, γ' are connected by a path in $\Omega \overline{\text{Ell}}_M$ (that is, if they are homotopic as free loops).

The spectral flow is a universal homotopy invariant for loops in the space $\mathcal{RF}^{\text{sa}}(H)$ of regular self-adjoint Fredholm operators on a separable complex Hilbert space H . But the space $\overline{\text{Ell}}(E)$ is only a very small part of $\mathcal{RF}^{\text{sa}}(L^2(M; E))$. We cannot expect a priori from the spectral flow to be a universal invariant for properties (Po-P2). However, the following theorem shows that this is indeed the case:

Theorem 2. *A function Φ from $\Omega \overline{\text{Ell}}_M$ to a commutative group Λ satisfies properties (Po-P2) if and only if it has the form $\Phi(\gamma) = \text{sf}(\gamma) \cdot \lambda$ for some constant $\lambda \in \Lambda$.*

We postpone the proof of this theorem to Section 9.

Remark. It may seem odd that we define $\overline{\text{Ell}}_M$ as the disjoint union of $\overline{\text{Ell}}(2n_M)$ rather than the disjoint union of $\overline{\text{Ell}}(E)$ with E running over isomorphism classes of Hermitian vector bundles over M . The reason is that we want to impose as weak conditions on Φ as possible, and in particular avoid the requirement of invariance of Φ with respect to the conjugation by unitary automorphisms of E . As Theorem 2 shows, properties (Po-P2) per se are strong enough to define Φ up to a multiplicative constant.

Paths with conjugate ends. Recall that for a topological space X and a homeomorphism g of X we denoted by $\Omega_g X$ the space of continuous paths $\gamma: [0, 1] \rightarrow X$ such that $\gamma(1) =$

$g\gamma(o)$, with the compact-open topology. Identifying S^1 with the factor $[0, 1]/\{0, 1\}$, we obtain the natural identification of $\Omega_{\text{Id}}X$ with ΩX .

Let G be a (discrete) group acting on a topological space X ; we denote by $\Omega^G X$ the disjoint union of spaces $\Omega_g X$ with g running through G . We write (x, g) for an element $x \in \Omega_g X$ considered as an element of the union $\Omega^G X$.

The group $\mathcal{U}(E)$ of smooth unitary bundle automorphisms (considered as a discrete group) continuously acts on $\overline{\text{Ell}}(E)$, so $\Omega^{\mathcal{U}(E)} \overline{\text{Ell}}(E)$ is defined. We will write $\Omega^{\mathcal{U}} \overline{\text{Ell}}(E)$ instead of $\Omega^{\mathcal{U}(E)} \overline{\text{Ell}}(E)$ for brevity.

Denote by $\Omega^{\mathcal{U}} \overline{\text{Ell}}_M$ the disjoint union of $\Omega^{\mathcal{U}} \overline{\text{Ell}}(2k_M)$ for all integer $k \geq 0$. The space $\Omega^{\mathcal{U}} \overline{\text{Ell}}_M$ has the natural structure of graded topological monoid, with the monoid operation given by the pointwise direct sum $\Omega_g \overline{\text{Ell}}(2k_M) \oplus \Omega_{g'} \overline{\text{Ell}}(2k'_M) \hookrightarrow \Omega_{g \oplus g'} \overline{\text{Ell}}((2k + 2k')_M)$. We define the submonoid $\Omega^{\mathcal{U}} \overline{\text{Ell}}_M^o$ of $\Omega^{\mathcal{U}} \overline{\text{Ell}}_M$ in the same manner. The monoid $\Omega \overline{\text{Ell}}_M$, resp. $\Omega \overline{\text{Ell}}_M^o$ considered in the previous subsection is a submonoid of $\Omega^{\mathcal{U}} \overline{\text{Ell}}_M$, resp. $\Omega^{\mathcal{U}} \overline{\text{Ell}}_M^o$.

Suppose that we have a function Φ which maps $\Omega^{\mathcal{U}} \overline{\text{Ell}}_M$ to some commutative group Λ . We are interested in functions Φ satisfying the following three properties:

(Po^U) Zero crossing. $\Phi(\gamma, g) = 0$ for every $(\gamma, g) \in \Omega^{\mathcal{U}} \overline{\text{Ell}}_M^o$.

(P1^U) Additivity with respect to direct sum. $\Phi(\gamma \oplus \gamma', g \oplus g') = \Phi(\gamma, g) + \Phi(\gamma', g')$ for every $(\gamma, g), (\gamma', g') \in \Omega^{\mathcal{U}} \overline{\text{Ell}}_M$.

(P2^U) Homotopy invariance. For every k and g , if γ, γ' are connected by a path in $\Omega_g \overline{\text{Ell}}(2k_M)$, then $\Phi(\gamma, g) = \Phi(\gamma', g)$.

Theorem 3. *A function Φ from $\Omega^{\mathcal{U}} \overline{\text{Ell}}_M$ to a commutative group Λ satisfies properties (Po^U–P2^U) if and only if it has the form $\Phi(\gamma, g) = \text{sf}(\gamma) \cdot \lambda$ for some constant $\lambda \in \Lambda$.*

We postpone the proof of this theorem to Section 9.

Symbolic universality of the spectral flow. Recall that $A \in \text{Ell}(E)$ is called a Dirac operator if $\sigma_A(\xi)^2 = \|\xi\|^2 \text{Id}_E$ for all $\xi \in T^*M$. We denote by $\text{Dir}(E)$ the subspace of $\text{Ell}(E)$ consisting of all *odd* Dirac operators, that is, operators having the form $D = \begin{pmatrix} 0 & D^- \\ D^+ & 0 \end{pmatrix}$ with respect to the chiral decomposition $E = E^+(D) \oplus E^-(D)$.

Denote by $\overline{\text{Dir}}(E)$ the subspace of $\overline{\text{Ell}}(E)$ consisting of all pairs (D, L) such that $D \in \text{Dir}(E)$ and L is given by formula (1.3) with a *unitary* self-adjoint automorphism T .

We consider three special subspaces of $\overline{\text{Dir}}(E)$. $\overline{\text{Dir}}_+(E)$ is the subspace of all $(D, L) \in \overline{\text{Dir}}(E)$ with L given by the formula

$$L = \{u^+ \oplus u^- \in E_0^+(D) \oplus E_0^-(D) : i\sigma(n)u^+ = u^-\}.$$

$\overline{\text{Dir}}_-(E)$ is the subspace of all $(D, L) \in \overline{\text{Dir}}(E)$ with L given by the formula

$$L = \{u^+ \oplus u^- \in E_\partial^+(D) \oplus E_\partial^-(D) : i\sigma(n)u^+ = -u^-\}.$$

Finally, $\overline{\text{Dir}}_\pm(E)$ is the subspace of $\overline{\text{Dir}}(E)$ consisting of all operators (D, L) that can be written in the form $(D', L') \oplus (D'', L'')$ with $(D', L') \in \overline{\text{Dir}}_+(E')$, $(D'', L'') \in \overline{\text{Dir}}_-(E'')$ for some orthogonal decomposition $E \cong E' \oplus E''$.

We can replace properties (Po) and $(\text{Po})^u$ by the following symbolic variants:

- $(\widetilde{\text{Po}})$ Φ is vanishing on constant loops in $\overline{\text{Dir}}(2k_M)$ and on loops in $\overline{\text{Dir}}_\pm(2k_M)$ for every k .
- $(\widetilde{\text{Po}})^u$ Φ is vanishing on constant loops in $\overline{\text{Dir}}(2k_M)$, on $\Omega^u \overline{\text{Dir}}_+(2k_M)$, and on $\Omega^u \overline{\text{Dir}}_-(2k_M)$ for every k .

Theorem 4. *A function Φ from $\Omega \overline{\text{Ell}}_M$ to a commutative group Λ satisfies properties $(\widetilde{\text{Po}})$, (P1) , and (P2) if and only if it has the form $\Phi(\gamma) = \text{sf}(\gamma) \cdot \lambda$ for some constant $\lambda \in \Lambda$.*

Theorem 5. *A function Φ from $\Omega^u \overline{\text{Ell}}_M$ to a commutative group Λ satisfies properties $(\widetilde{\text{Po}})^u$, $(\text{P1})^u$, and $(\text{P2})^u$ if and only if it has the form $\Phi(\gamma, g) = \text{sf}(\gamma) \cdot \lambda$ for some constant $\lambda \in \Lambda$.*

7 Auxiliary theorems

The proofs of Theorems 1–5 are based on the following two auxiliary theorems.

Theorem 6. *If a function Φ from $\Omega \overline{\text{Ell}}_M$ to a commutative group Λ satisfies properties $(\widetilde{\text{Po}})$, (P1) , and (P2) , then it has the form*

$$(7.1) \quad \Phi(\gamma) = c_1(\mathcal{F}(\gamma))[\partial M \times S^1] \cdot \lambda$$

for some constant $\lambda \in \Lambda$.

Theorem 7. *If a function Φ from $\Omega^u \overline{\text{Ell}}_M$ to a commutative group Λ satisfies properties $(\widetilde{\text{Po}})^u$, $(\text{P1})^u$, and $(\text{P2})^u$, then it has the form*

$$(7.2) \quad \Phi(\gamma, g) = c_1(\mathcal{F}(\gamma, g))[\partial M \times S^1] \cdot \lambda$$

for some constant $\lambda \in \Lambda$.

The rest of the section is devoted to the proof of these two theorems.

Continuity

For a smooth compact manifold N and a smooth fiber bundle V over N , we denote by $C^{\infty,s}(N;V)$ the space of smooth sections of V with C^s -topology, that is, the topology induced by the embedding $C^{\infty}(N;V) \hookrightarrow C^s(N;V)$.

Denote by $\Sigma(E)$ the set of all smooth bundle morphisms $\sigma: T^*M \rightarrow \text{End}(E)$ such that σ is a symbol of a formally self-adjoint elliptic operator. Equip $\Sigma(E)$ with the topology induced by the inclusion $\Sigma(E) \hookrightarrow C^1(M; TM \otimes \text{End}(E))$. Then the natural projection $\text{Ell}(E) \rightarrow \Sigma(E)$ is continuous, as well as the map $\Sigma(E) \rightarrow C^1(\partial M; \text{End}(E_{\partial}))$, $\sigma \mapsto \sigma(n)$.

Let e_1, e_2 be global sections of T^*M such that $(e_1(x), e_2(x))$ is a positive oriented orthonormal basis of T_x^*M for any $x \in M$.

Lemma 7.1. *The following maps are continuous:*

1. $Q: \Sigma(E) \rightarrow C^{\infty,1}(M; \text{End}(E))$, $\sigma \mapsto Q = \sigma(e_1)^{-1}\sigma(e_2)$;
2. $E^+, E^-: \Sigma(E) \rightarrow C^{\infty,1}(M; \text{Gr}(E))$;
3. $F: \overline{\text{Ell}}(E) \rightarrow C^{\infty,1}(\partial M; \text{Gr}(E_{\partial}))$.

Proof. 1. The maps from $\Sigma(E)$ to $C^{\infty,1}(M; \text{End}(E))$ taking σ to $\sigma(e_i)$, $i = 1, 2$, are continuous, so Q is also continuous.

2. The invariant subspace E_x^- of Q_x spanned by the generalized eigenspaces of Q_x corresponding to eigenvalues with negative imaginary part is an analytic function of Q_x and hence an analytic function of σ_x . Therefore, $E^-(\sigma)$ is a smooth subbundle of E for smooth σ , and the map $E^-: \Sigma(E) \rightarrow C^{\infty,1}(M; \text{Gr}(E))$ is continuous. The same is true for $E^+: \Sigma(E) \rightarrow C^{\infty,1}(M; \text{Gr}(E))$.

3. Denote by $\text{Gr}^{(2)}(E)$ the smooth subbundle of $\text{Gr}(E) \times_M \text{Gr}(E)$ whose fiber over $x \in M$ consists of pairs (V_x, W_x) of subspaces of E_x such that $E_x = V_x \oplus W_x$. For a smooth section (V, W) of $\text{Gr}^{(2)}(E)$ the projection $P_{V,W}$ of E on V along W is a smooth section of $\text{End}(E)$. The map $C^{\infty,1}(M; \text{Gr}^{(2)}(E)) \rightarrow C^{\infty,1}(M; \text{End}(E))$, $(V, W) \mapsto P_{V,W}$ is continuous. The same is true if we replace M by ∂M and E by E_{∂} . Therefore, the composition

$$\Sigma(E) \rightarrow C^{\infty,1}(M; \text{Gr}^{(2)}(E)) \rightarrow C^{\infty,1}(\partial M; \text{Gr}^{(2)}(E_{\partial})) \rightarrow C^{\infty,1}(\partial M; \text{End}(E_{\partial})),$$

$\sigma \mapsto (E^+, E^-) \mapsto (E_{\partial}^+, E_{\partial}^-) \mapsto P_{E_{\partial}^+, E_{\partial}^-}$ is continuous. Similarly, the following two maps are continuous:

$$\begin{aligned} \Sigma(E) &\rightarrow C^{\infty,1}(\partial M; \text{End}(E_{\partial})), & \sigma &\mapsto P_{E_{\partial}^-, (E_{\partial}^-)^{\perp}}, \\ \overline{\text{Ell}}(E) &\rightarrow C^{\infty,1}(\partial M; \text{End}(E_{\partial})), & (A, L) &\mapsto P_{L, E_{\partial}^+}. \end{aligned}$$

Taking the composition of these maps, we obtain the continuity of the map

$$T': \overline{\text{Ell}}(E) \rightarrow C^{\infty,1}(\partial M; \text{End}(E_\partial)), \quad T'(A, L) = P_{E_\partial^-, (E_\partial^-)^\perp} \circ \sigma(n) \circ P_{E_\partial^+, E_\partial^-} P_{L, E_\partial^+} + P_{E_\partial^+, E_\partial^-}.$$

(We consider here the auxiliary automorphism T' instead of T by technical reasons: the problem with T is that T takes values in the space $C^\infty(\partial M; \text{End}(E_\partial^-))$ which depends on σ .) Denote by χ_S the characteristic function of a subset S of \mathbb{R} . The simple check shows that $T'|_{E_\partial^+} = \text{Id}$, $T'|_{E_\partial^-} = T$, so $\chi_{(-\infty, 0)}(T_x) = \chi_{(-\infty, 0)}(T'_x)$. Hence F_x considered as a point of $\text{Gr}(E_x)$ coincides with $\text{Im}(\chi_{(-\infty, 0)}(T'_x))$ and thus is an analytic function of T'_x . Therefore, F is a smooth subbundle of E and continuously depends on T' in C^1 -metric. Together with the continuity of T' this implies the continuity of the map $F: \overline{\text{Ell}}(E) \rightarrow C^{\infty,1}(M; \text{Gr}(E_\partial))$. This completes the proof of the lemma. \square

Deformation retraction

The third part of Lemma 7.1 implies that the right-hand sides of (7.1) and (7.2) do not change at the continuous deformation of γ in $\Omega \overline{\text{Ell}}(E)$. On the other hand, the left hand sides of (7.1) and (7.2) do not change at such deformation due to properties (P2) and (P2^U). Therefore we can deform γ as we want in the course of the proof. In particular, we can replace $\overline{\text{Ell}}(E)$ by its deformation retract. Our next goal is to prove the following fact.

Lemma 7.2. *There exists a deformation retraction of $\overline{\text{Ell}}(E)$ onto a subspace of $\overline{\text{Dir}}(E)$ (that is, a continuous map $h: [0, 1] \times \overline{\text{Ell}}(E) \rightarrow \overline{\text{Ell}}(E)$ such that $h_0 = \text{Id}_{\overline{\text{Ell}}(E)}$, $\text{Im } h_1 \subset \overline{\text{Dir}}(E)$, and the restriction of h_1 on $\text{Im } h_1$ is the identity). Moreover, h can be chosen leaving invariant both $E^-(A)$ and $F(A, L)$ and “almost $U(E)$ -equivariant”. By the last we mean that, for every $g \in U(E)$ and $s \in [0, 1]$, for $(A', L') = gh_s(A, L)$ and $(A'', L'') = h_s(g(A, L))$ we have $L' = L''$ and the symbols of operators A' and A'' coincide.*

To prove it, we need some preparation, so we postpone the proof till the end of the subsection.

Lemma 7.3. *The map $p: \text{Ell}(E) \rightarrow \Sigma(E)$ is surjective and has a continuous section $r: \Sigma(E) \rightarrow \text{Ell}(E)$ such that $r \circ p$ is fiberwise homotopic to the identity map.*

Proof. We define a section $r: \Sigma(E) \rightarrow \text{Ell}(E)$ by the formula $r(\sigma) = (\sigma_1 \nabla_1 + \sigma_2 \nabla_2) / 2 + (\sigma_1 \nabla_1 + \sigma_2 \nabla_2)^t / 2$, where $\sigma_i = \sigma(e_i)$, ∇ is some connection on E and superscript t means taking of formally adjoint operator. The operation of taking formally adjoint operator leaves invariant symbol. Moreover, it defines a continuous transformation of the space of first order operators with the topology defined by the inclusion to $C^1(M; \text{End}(E))^2 \times C^0(M; \text{End}(E))$, $\sigma_1 \nabla_1 + \sigma_2 \nabla_2 + a \mapsto (\sigma_1, \sigma_2, a)$. Thus r is a continuous section of p and defines a trivialization of the vector bundle $\text{Ell}(E) \rightarrow \Sigma(E)$ with the

fiber $C^{\infty,0}(M; \text{End}^{\text{sa}}(E))$. Since the fiber is a vector space, $r \circ p$ is fiberwise homotopic to the identity map. This completes the proof of the lemma. \square

Denote by $\Sigma'(E)$ the subspace of $\Sigma(E)$ consisting of symbols of Dirac operators.

Lemma 7.4. *The restriction of p to $\text{Dir}(E)$ has a continuous section $r': \Sigma'(E) \rightarrow \text{Dir}(E)$.*

Proof. Let $\sigma \in \Sigma'(E)$ and $D = r(\sigma)$. Denote by S the bundle automorphism of E , whose restrictions on the fibers are the orthogonal reflections in the fibers of $E^-(\sigma)$. Then the Dirac operator $r'(\sigma) = (D - SDS)/2$ is odd with respect to the chiral decomposition $E = E^+(\sigma) \oplus E^-(\sigma)$ and has the same symbol σ as D . Since S depends continuously on σ , the map $r': \Sigma'(E) \rightarrow \text{Dir}(E)$ is a continuous section of $p|_{\text{Dir}(E)}$. This completes the proof of the lemma. \square

Lemma 7.5. *The subspace $\Sigma'(E)$ is a strong deformation retract of $\Sigma(E)$. Moreover, a deformation retraction can be chosen $U(E)$ -equivariant and leaving invariant $E^-(\sigma)$.*

Proof. For any $\sigma \in \Sigma(E)$ the automorphism $Q = \sigma(e_1)^{-1}\sigma(e_2)$ of E leaves the subbundles $E^- = E^-(\sigma)$, $E^+ = E^+(\sigma)$ invariant; we denote by Q^- (resp. Q^+) the restriction of Q to E^- (resp. E^+). By the construction of E^\pm , all eigenvalues of Q^- (resp. Q^+) on every fiber of E^- (resp. E^+) have negative (resp. positive) imaginary part.

Denote by J the restriction of $\sigma(e_1)$ to E^- ; it is a smooth bundle isomorphism from E^- onto its orthogonal complement $(E^-)^\perp$.

Finally, with every $\sigma \in \Sigma(E)$ we associate the quadruple

$$(7.3) \quad \vartheta(\sigma) = (E^-, E^+, J, Q^-).$$

Denote by $\Theta(E)$ the set of all quadruples (E^-, E^+, J, Q^-) such that E^-, E^+ are transversal smooth subbundles of E of half rank (that is, $\text{rank } E^- = \text{rank } E^+ = \frac{1}{2} \text{rank } E$), J is a smooth bundle isomorphism of E^- onto $(E^-)^\perp$, and Q^- is a smooth bundle automorphism of E^- such that all eigenvalues of Q^- have negative imaginary part for every $x \in M$.

Equip $\Theta(E)$ with the topology induced by the inclusion

$$\Theta(E) \hookrightarrow C^1(M; \text{Gr}(E))^2 \times C^1(M; \text{End}(E))^2, \quad (E^-, E^+, J, Q^-) \mapsto (E^-, E^+, J \oplus 0_{E^+}, Q^- \oplus 0_{E^+}).$$

Claim. *The map (7.3) defines a homeomorphism between the spaces $\Sigma(E)$ and $\Theta(E)$.*

Proof of the claim. Let us show that ϑ is a bijection. Let $(E^-, E^+, J, Q^-) \in \Theta(E)$. Then $\sigma_1^- = J$, $\sigma_2^- = JQ^-$ are smooth bundle isomorphisms from E^- onto $(E^-)^\perp$.

The Hermitian structure on E defines the non-degenerate pairings $E_x^+ \times (E_x^-)^\perp \rightarrow \mathbb{C}$ and $(E_x^+)^\perp \times E_x^- \rightarrow \mathbb{C}$ for each $x \in M$. Hence there exist (unique) smooth bundle isomorphisms σ_1^+ , σ_2^+ from E^+ onto $(E^+)^\perp$ such that $\langle \sigma_i^+ u, v \rangle = \langle u, \sigma_i^- v \rangle$ for any $u \in E_x^+$, $v \in E_x^-$, $x \in M$. We define the endomorphism σ_i of E by the condition that the restriction of σ_i to E^+ , resp. E^- coincides with σ_i^+ , resp. σ_i^- .

Every elements $u, v \in E_x$ can be written as $u = u^+ + u^-$, $v = v^+ + v^-$ with $u^+, v^+ \in E_x^+$, $u^-, v^- \in E_x^-$. We have $\langle \sigma_i u, v \rangle = \langle \sigma_i^+ u^+, v^- \rangle + \langle \sigma_i^- u^-, v^+ \rangle = \langle u^+, \sigma_i^- v^- \rangle + \langle u^-, \sigma_i^+ v^+ \rangle = \langle u, \sigma_i v \rangle$. Thus σ_1, σ_2 are self-adjoint.

Let $(c_1, c_2) \in \mathbb{R}^2 \setminus \{0\}$. Then $c_1 \sigma_1^- + c_2 \sigma_2^- = \sigma_1^-(c_1 + c_2 Q^-)$ is an isomorphism of E^- onto $(E^-)^\perp$. By definition of σ_i^+ , $\langle (c_1 \sigma_1^+ + c_2 \sigma_2^+) u, v \rangle = \langle u, (c_1 \sigma_1^- + c_2 \sigma_2^-) v \rangle$ for any $u \in E_x^+$, $v \in E_x^-$. Therefore, $c_1 \sigma_1^+ + c_2 \sigma_2^+$ is an isomorphism of E^+ onto $(E^+)^\perp$. The direct sum decompositions $E^- \oplus E^+ = E = (E^-)^\perp \oplus (E^+)^\perp$ imply that $c_1 \sigma_1 + c_2 \sigma_2$ is a smooth bundle automorphism of E . Thus (σ_1, σ_2) determines the self-adjoint elliptic symbol $\sigma \in \Sigma(E)$, $\sigma(e_i) = \sigma_i$.

The automorphism $Q = \sigma_1^{-1} \sigma_2$ of E leaves E^- , E^+ invariant, and the restriction of Q to E^- coincides with Q^- . All eigenvalues of Q^- have negative imaginary part. Ranks of E^- and E^+ coincide, so by Proposition 3.1 all eigenvalues of the restriction of Q to E^+ have positive imaginary part.

By construction, $\vartheta(\sigma) = (E^-, E^+, J, Q^-)$. The same construction shows that σ is determined uniquely by the quadruple (E^-, E^+, J, Q^-) . Therefore ϑ defines a bijection between $\Sigma(E)$ and $\Theta(E)$.

By Lemma 7.1, ϑ is continuous. The construction of the inverse map given above shows that ϑ^{-1} is also continuous. This completes the proof of the claim. \square

By this claim, instead of a deformation retraction of $\Sigma(E)$ we can construct a deformation retraction of $\Theta(E)$ onto the subspace

$$\Theta'(E) = \vartheta(\Sigma'(E)) = \left\{ (E^-, E^+, J, Q^-) \in \Theta(E) : E^+ = (E^-)^\perp, J \in \mathcal{U}(E^-, E^+), Q^- = -i \text{Id} \right\}.$$

For fixed E^- , all three ingredients of the triple (E^+, J, Q^-) can be deformed independently of one another. We define a homotopy $h_s(E^-, E^+, J, Q^-) = (E^-, E_s^+, J_s, Q_s^-)$ by the formulas

$$J_s = \left(s(JJ^*)^{-1/2} + 1 - s \right) J, \quad Q_s^- = -is \text{Id} + (1 - s)Q^-,$$

and E_s^+ be the graph of $(1 - s)B$, where B is the smooth homomorphism from $(E^-)^\perp$ to E^- with the graph E^+ .

Obviously, $h_0 = \text{Id}$, the image of h_1 is contained in $\Theta'(E)$, and the restriction of h_s to $\Theta'(E)$ is the identity for all $s \in [0, 1]$. Thus h defines a deformation retraction of $\Sigma(E)$ onto $\Sigma'(E)$. By construction, h is $\mathcal{U}(E)$ -equivariant. This completes the proof of Lemma 7.5. \square

Let

$$\overline{\Sigma}(E) = \left\{ (\sigma_A, L) : (A, L) \in \overline{\text{Ell}}(E) \right\}$$

with the topology induced by the inclusion $\overline{\Sigma}(E) \hookrightarrow \Sigma(E) \times C^1(\partial M; \text{Gr}(E_\partial))$. The natural projection $\bar{p} : \overline{\text{Ell}}(E) \rightarrow \overline{\Sigma}(E)$, $(A, L) \mapsto (\sigma_A, L)$, is continuous and $\mathcal{U}(E)$ -equivariant.

Lemma 7.6. *The subspace $\bar{\Sigma}'(E) = \bar{p}(\overline{\text{Dir}}(E))$ is a strong deformation retract of $\bar{\Sigma}(E)$, and a deformation retraction can be chosen $\mathcal{U}(E)$ -equivariant and leaving invariant both $E^-(\sigma)$ and $F(\sigma, L)$.*

Proof. The homotopy h constructed in the previous lemma leaves invariant E^- . Thus we can define a strong deformation retraction of $\bar{\Sigma}(E)$ onto $\bar{\Sigma}'(E)$ by the formula

$$\bar{h}_s(\sigma, T) = (h_s(\sigma), (1 - s + s|T|^{-1})T)$$

(recall that we can write an element of $\bar{\Sigma}(E)$ as (σ, T) instead of (σ, L) , where L and T correspond one another as described in Proposition 5.1). Obviously, \bar{h}_s is $\mathcal{U}(E)$ -equivariant and leaves invariant both $E^-(\sigma)$ and $F(\sigma, L)$. This completes the proof of the lemma. \square

Proof of Lemma 7.2. Throughout the proof, we use for brevity the word “nice” as a substitute for “leaves invariant both $E^-(A)$ and $F(A, L)$ and is almost $\mathcal{U}(E)$ -equivariant”.

Define the sections $\bar{r}: \bar{\Sigma}(E) \rightarrow \overline{\text{Ell}}(E)$ and $\bar{r}': \bar{\Sigma}'(E) \rightarrow \overline{\text{Dir}}(E)$ by the formulas $\bar{r}(A, L) = (r(A), L)$ and $\bar{r}'(D, L) = (r'(D), L)$, where r is the section from Lemma 7.3 and r' is the section from Lemma 7.4. The fiberwise homotopy from Lemma 7.3 between $r \circ p$ and $\text{Id}_{\overline{\text{Ell}}(E)}$ defines a fiberwise homotopy between $\bar{r} \circ \bar{p}$ and $\text{Id}_{\overline{\text{Ell}}(E)}$ and thus gives a nice deformation retraction of $\overline{\text{Ell}}(E)$ onto $\bar{r}(\bar{\Sigma}(E))$. Let \bar{h}_s be the deformation retraction of $\bar{\Sigma}(E)$ onto $\bar{\Sigma}'(E)$ from Lemma 7.6. Then $\bar{r} \circ \bar{h}_s \circ \bar{p}$ is a nice deformation retraction of $\bar{r}(\bar{\Sigma}(E))$ onto $\bar{r}(\bar{\Sigma}'(E))$. Combining these two nice retractions $\overline{\text{Ell}}(E) \rightarrow \bar{r}(\bar{\Sigma}(E))$ and $\bar{r}(\bar{\Sigma}(E)) \rightarrow \bar{r}(\bar{\Sigma}'(E))$, we obtain a nice deformation retraction q_s of $\overline{\text{Ell}}(E)$ onto $\bar{r}(\bar{\Sigma}'(E)) \subset \bar{p}^{-1}(\bar{\Sigma}'(E))$.

$$\begin{array}{ccccc}
 & & \xleftarrow{q'_1} & & \xleftarrow{q_1} \\
 \overline{\text{Dir}}(E) & \xrightarrow{\quad} & \bar{p}^{-1}(\bar{\Sigma}'(E)) & \xrightarrow{\quad} & \overline{\text{Ell}}(E) \\
 & \searrow & \downarrow \bar{p} & \nearrow \bar{r} & \downarrow \bar{p} \\
 & & \bar{\Sigma}'(E) & \xrightarrow{\quad} & \bar{\Sigma}(E) \\
 & \nearrow \bar{r}' & & \nwarrow \bar{h}_1 & \\
 & & & &
 \end{array}$$

We define a fiberwise deformation retraction of $\bar{p}^{-1}(\bar{\Sigma}'(E))$ onto $\overline{\text{Dir}}(E)$ by the formula $q'_s(D, L) = ((1 - s)D + sr'p(D), L)$. Combining q and q' , we obtain a nice deformation retraction h of $\overline{\text{Ell}}(E)$ onto the subspace $\bar{r}'(\bar{\Sigma}'(E))$ of $\overline{\text{Dir}}(E)$,

$$h_s(A, L) = \begin{cases} q_{2s}(A, L) & \text{for } 0 \leq s \leq 1/2, \\ q'_{2s-1}q_1(A, L) & \text{for } 1/2 \leq s \leq 1. \end{cases}$$

This completes the proof of the lemma. \square

Proof of Theorem 6

For $g \in \mathcal{U}(E)$, we denote by $\mathcal{E}(g)$ the vector bundle over $M \times S^1$ obtained from E and g in the same manner as $\mathcal{E}_\partial = \mathcal{E}_\partial(g)$ was obtained from E_∂ and g in Section 5. For $\gamma \in \Omega_g \overline{\text{Ell}}(E)$, we denote by $\mathcal{E}^-(\gamma, g)$ the subbundle of $\mathcal{E}(g)$ obtained from the one-parameter family $E^-(\gamma(t))$ and g in the same manner as $\mathcal{F}(\gamma, g)$ was obtained from the family $F(\gamma(t))$ and g in Section 5. We denote by $\mathcal{E}_\partial^-(\gamma, g)$ the restriction of $\mathcal{E}^-(\gamma, g)$ to $\partial M \times S^1$. We will write $\mathcal{E}^-(\gamma)$, resp. $\mathcal{E}_\partial^-(\gamma)$ instead of $\mathcal{E}^-(\gamma, \text{Id})$, resp. $\mathcal{E}_\partial^-(\gamma, \text{Id})$.

Lemma 7.7. *There exists a deformation retraction of $\Omega \overline{\text{Ell}}(E)$ onto a subspace of $\Omega \overline{\text{Dir}}(E)$ leaving invariant both $\mathcal{E}^-(\gamma)$ and $\mathcal{F}(\gamma)$.*

Proof. This is an immediate corollary of Lemma 7.2. \square

Suppose now that $\Phi: \Omega \overline{\text{Ell}}_M \rightarrow \Lambda$ satisfies properties $(\widetilde{\text{Po}})$, (P_1) , and (P_2) .

Lemma 7.8. *Let $\gamma \in \Omega \overline{\text{Ell}}_M$. If $\mathcal{F}(\gamma) = 0$ or $\mathcal{F}(\gamma) = \mathcal{E}_\partial^-(\gamma)$, then $\Phi(\gamma) = 0$.*

Proof. By Lemma 7.7 and Property (P_2) , we can suppose without loss of generality that $\gamma \in \Omega \overline{\text{Dir}}(2k_M)$. If $\mathcal{F}(\gamma) = 0$, then $\gamma \in \Omega \overline{\text{Dir}}_+(2k_M)$. If $\mathcal{F}(\gamma) = \mathcal{E}_\partial^-(\gamma)$, then $\gamma \in \Omega \overline{\text{Dir}}_-(2k_M)$. In both cases Property $(\widetilde{\text{Po}})$ implies $\Phi(\gamma) = 0$. \square

Lemma 7.9. *There is (unique) homomorphism $\varphi: K^0(\partial M \times S^1) \rightarrow \Lambda$ such that $\Phi(\gamma) = \varphi[\mathcal{F}(\gamma)]$ for every $\gamma \in \Omega \overline{\text{Ell}}_M$.*

Proof. Let us show first that $\Phi(\gamma)$ depends only on the isomorphism class of $\mathcal{F}(\gamma)$. Indeed, suppose that $\mathcal{F}(\gamma_1)$ is isomorphic (as a vector bundle) to $\mathcal{F}(\gamma_2)$ for $\gamma_i \in \Omega \overline{\text{Ell}}(E_i)$, $i = 1, 2$. By Lemma 7.7 and Property (P_2) , we can suppose without loss of generality that $\gamma_i \in \Omega \overline{\text{Dir}}(E_i)$. For sufficiently large integer k , the vector subbundles $\mathcal{F}(\gamma_1) \oplus 0 \oplus 0$ and $0 \oplus \mathcal{F}(\gamma_2) \oplus 0$ of $\mathcal{E}_\partial^-(\gamma_1) \oplus \mathcal{E}_\partial^-(\gamma_2) \oplus k_{\partial M \times S^1}$ are homotopic, and the homotopy h_s can be chosen smooth by $x \in M$ and C^1 -continuous by $s \in [0, 1]$, $t \in S^1$. Let $\gamma_i^+ \in \Omega \overline{\text{Dir}}_+(E_i)$ be the loop obtained from γ_i by replacing all T_t by the identity. Let δ be an arbitrary constant loop in $\overline{\text{Dir}}_+(2k_M)$. Then the loops $\gamma_1 \oplus \gamma_2^+ \oplus \delta$ and $\gamma_1^+ \oplus \gamma_2 \oplus \delta$ are homotopic in $\overline{\text{Dir}}(E_1 \oplus E_2 \oplus 2k_M)$. By Properties (P_1) and (P_2) , $\Phi(\gamma_1) + \Phi(\gamma_2^+) + \Phi(\delta) = \Phi(\gamma_1^+) + \Phi(\gamma_2) + \Phi(\delta)$. Since $\mathcal{F}(\gamma_i^+) = 0$, Lemma 7.8 implies $\Phi(\gamma_i^+) = 0$. Therefore, $\Phi(\gamma_1) = \Phi(\gamma_2)$.

Every isomorphism class of vector bundles over $\partial M \times S^1$ can be represented by a smooth subbundle of a trivial vector bundle of sufficient large rank k , and so can be represented by $\mathcal{F}(\gamma)$ for some $\gamma \in \Omega \overline{\text{Dir}}(2k_M)$. The additivity of both Φ and \mathcal{F} with respect to direct sums of elements of $\Omega \overline{\text{Ell}}_M$ completes the proof of the lemma. \square

Lemma 7.10. *The homomorphism φ vanishes on the image of $\pi^*: K^0(\partial M) \rightarrow K^0(\partial M \times S^1)$, where π denotes the projection of $\partial M \times S^1$ to ∂M .*

Proof. It is sufficient to prove that $\varphi[\pi^*V] = 0$ for any smooth vector bundle V over ∂M . We can realize V as $F(D, L)$ for some $(D, L) \in \overline{\text{Dir}}_M$. Indeed, V can be embed as a smooth subbundle to a trivial vector bundle over ∂M of sufficiently large rank k . Define the symbol $\sigma \in \Sigma'(k_M \oplus k_M)$ by the formula

$$(7.4) \quad \sigma(e_1) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma(e_2) = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}.$$

Let $D = r'(\sigma)$ and $T = (-1)_V \oplus 1_{V^\perp}$, where V^\perp is the orthogonal complement of V in $E_\partial^-(D) = k_{\partial M}$. Then $(D, T) \in \overline{\text{Dir}}(k_M \oplus k_M)$ and $F(D, T) = V$ as a subbundle of $E_\partial^-(D)$. Applying property (Po) to the constant loop $\gamma \in \Omega \overline{\text{Dir}}(k_M \oplus k_M)$ defined by $\gamma(t) \equiv (D, T)$, we obtain $\Phi(\gamma) = 0$. But $\Phi(\gamma) = \varphi[\mathcal{F}(\gamma)] = \varphi[\pi^*V]$. Hence φ vanishes on the image of $K^0(\partial M)$. \square

Lemma 7.11. *The homomorphism φ vanishes on the image of $j^*: K^0(M \times S^1) \rightarrow K^0(\partial M \times S^1)$, where j denotes the embedding of $\partial M \times S^1$ to $M \times S^1$.*

Proof. It is sufficient to prove that $\varphi[j^*V] = 0$ for any smooth vector bundle V over $M \times S^1$. We can embed V as a smooth subbundle to a trivial vector bundle over $M \times S^1$ of sufficiently large rank k . Let (V_t) , $t \in S^1$, be the correspondent one-parameter family of subbundles of k_M , and let W_t be the orthogonal complement of V_t in k_M . Denote by R_t the bundle automorphism of k_M , whose restrictions on the fibers are the orthogonal reflections in the fibers of V_t . Denote by T_t the restriction of R_t to ∂M .

Define the symbol $\sigma \in \Sigma'(k_M \oplus k_M)$ by formula (7.4). Let $D = r'(\sigma)$ be the lifting of σ to $\text{Dir}(k_M \oplus k_M)$, and let $D_t = (D + S_t D S_t)/2$, where $S_t = R_t \oplus R_t \in \text{End}(k_M \oplus k_M)$. Then $(D_t, -T_t)$ is the direct sum of Dirac operators $(D'_t, -\text{Id}) \in \overline{\text{Dir}}_-(V_t \oplus V_t)$ and $(D''_t, \text{Id}) \in \overline{\text{Dir}}_+(W_t \oplus W_t)$ for the orthogonal decomposition $2k_M \cong (V_t \oplus V_t) \oplus (W_t \oplus W_t)$. Therefore, $(D_t, -T_t) \in \overline{\text{Dir}}_\pm(k_M \oplus k_M)$ for every $t \in S^1$. By Property (Po) , Φ vanishes on the loop γ , $\gamma(t) = (D_t, -T_t)$. Since $\mathcal{F}(\gamma) = j^*V$, we obtain $\varphi[j^*V] = 0$. This completes the proof of the lemma. \square

Lemma 7.12. *Let φ be a homomorphism from $K^0(\partial M \times S^1)$ to a commutative group Λ . Suppose that φ vanishes on the images of $\pi^*: K^0(\partial M) \rightarrow K^0(\partial M \times S^1)$ and $j^*: K^0(M \times S^1) \rightarrow K^0(\partial M \times S^1)$. Then φ has the form $\varphi[\mathcal{F}] = c_1(\mathcal{F})[\partial M \times S^1] \cdot \lambda$ for some constant $\lambda \in \Lambda$.*

Proof. The first Chern class is additive with respect to direct sum of vector bundles, so we can define the homomorphism of commutative groups $\psi: K^0(\partial M \times S^1) \rightarrow \mathbb{Z}$ such that $\psi[\mathcal{F}] = c_1(\mathcal{F})[\partial M \times S^1]$ for any vector bundle \mathcal{F} over $\partial M \times S^1$. The kernel of ψ can be written as the span of two subgroups of $K^0(\partial M \times S^1)$: $\text{Ker } \psi = \text{span}(\text{Im } \pi^* \cup \text{Im } j^*)$, so φ vanishes on the kernel of ψ . Since ψ is surjective, φ factors through ψ . Each homomorphism from \mathbb{Z} to Λ can be written as a multiplication by some constant $\lambda \in \Lambda$. Thus $\varphi[\mathcal{F}] = \psi[\mathcal{F}] \cdot \lambda = c_1(\mathcal{F})[\partial M \times S^1] \cdot \lambda$ for every vector bundle \mathcal{F} over $\partial M \times S^1$. This completes the proof of the lemma. \square

Now we are ready to complete the proof of Theorem 6. By Lemma 7.9, there is a homomorphism $\varphi: K^0(\partial M \times S^1) \rightarrow \Lambda$ such that $\Phi(\gamma) = \varphi[\mathcal{F}(\gamma)]$ for every $\gamma \in \Omega \overline{\text{Ell}}_M$. By Lemmas 7.10 and 7.11, φ vanishes on the images of π^* and j^* . By Lemma 7.12, φ has the form $\varphi[\mathcal{F}] = c_1(\mathcal{F})[\partial M \times S^1] \cdot \lambda$ for some constant $\lambda \in \Lambda$. This completes the proof of the theorem. \square

Proof of Theorem 7

The proof of this theorem is going along the same lines as the proof of Theorem 6, with some variations.

Lemma 7.13. *Let g be a unitary bundle automorphism of E . Then there exists a deformation retraction of $\Omega_g \overline{\text{Ell}}(E)$ onto a subspace of $\Omega_g \overline{\text{Dir}}(E)$ leaving invariant both $\mathcal{E}^-(\gamma, g)$ and $\mathcal{F}(\gamma, g)$.*

Proof. Let $\rho, \rho': [0, 1] \rightarrow \mathbb{R}$ be a partition of unity subordinated to the covering $[0, 1] = U \cup U'$, $U = [0, 2/3]$, $U' = [1/3, 1]$, that is, $\text{supp } \rho \subset U$, $\text{supp } \rho' \subset U'$, and $\rho + \rho' \equiv 1$. Let $h: [0, 1] \times \overline{\text{Ell}}(E) \rightarrow \overline{\text{Ell}}(E)$ be the deformation retraction of $\overline{\text{Ell}}(E)$ onto a subspace of $\overline{\text{Dir}}(E)$ constructed in Lemma 7.2. Then a desired deformation retraction $H: [0, 1] \times \Omega_g \overline{\text{Ell}}(E) \rightarrow \Omega_g \overline{\text{Ell}}(E)$ can be defined by the formula

$$H_s(\gamma)(t) = (\rho(t)A_{s,t} + \rho'(t)A'_{s,t}, L_{s,t}),$$

where $(A_{s,t}, L_{s,t}) = h_s(\gamma(t))$ and $(A'_{s,t}, L_{s,t}) = gh_s(g^{-1}\gamma(t))$. Indeed, h_s is almost $U(E)$ -equivariant, so the operators $A_{s,t}$ and $A'_{s,t}$ have the same symbols, and their linear combination $\rho(t)A_{s,t} + \rho'(t)A'_{s,t}$ lies in $\text{Ell}(E)$. The symbols and the chiral decompositions of the odd Dirac operators $A_{1,t}$ and $A'_{1,t}$ coincide, so their linear combination $\rho(t)A_{1,t} + \rho'(t)A'_{1,t}$ lies in $\text{Dir}(E)$, thus $H_1(\gamma)(t) \in \overline{\text{Dir}}(E)$. For $s = 0$ we have $(A'_{0,t}, L_{0,t}) = gh_0(g^{-1}\gamma(t)) = \gamma(t) = (A_{0,t}, L_{0,t})$, so $\rho(t)A_{0,t} + \rho'(t)A'_{0,t} = (\rho(t) + \rho'(t))A_{0,t} = A_{0,t}$, thus $H_0(\gamma)(t) = \gamma(t)$. For any $s \in [0, 1]$ we have

$$H_s(\gamma)(1) = (A'_{s,1}, L_{s,1}) = gh_s(g^{-1}\gamma(1)) = gh_s(\gamma(0)) = g(A_{s,0}, L_{s,0}) = gH_s(\gamma)(0),$$

so $H_s(\gamma) \in \Omega_g \overline{\text{Ell}}(E)$. Since h_s leaves invariant $E^-(A)$ and $F(A, L)$, H_s leaves invariant $\mathcal{E}^-(\gamma, g)$ and $\mathcal{F}(\gamma, g)$. This completes the proof of the lemma. \square

Suppose now that $\Phi: \Omega \overline{\text{Ell}}_M \rightarrow \Lambda$ satisfies properties $(\widetilde{\text{Po}}^U)$, (P_1^U) , and (P_2^U) .

Lemma 7.14. *Let $(\gamma, g) \in \Omega^U \overline{\text{Ell}}_M$. If $\mathcal{F}(\gamma, g) = 0$ or $\mathcal{F}(\gamma, g) = \mathcal{E}_\partial^-(\gamma, g)$, then $\Phi(\gamma, g) = 0$.*

Proof. By Lemma 7.13 and Property (P_2^U) , we can suppose without loss of generality that $\gamma \in \Omega_g \overline{\text{Dir}}(2k_M)$. If $\mathcal{F}(\gamma, g) = 0$, then $\gamma \in \Omega_g \overline{\text{Dir}}_+(2k_M)$. If $\mathcal{F}(\gamma, g) = \mathcal{E}_\partial^-(\gamma, g)$, then $\gamma \in \Omega_g \overline{\text{Dir}}_-(2k_M)$. In both cases Property $(\widetilde{\text{Po}}^U)$ implies $\Phi(\gamma, g) = 0$. \square

Lemma 7.15. *There is (unique) homomorphism $\varphi: K^0(\partial M \times S^1) \rightarrow \Lambda$ such that $\Phi(\gamma, g) = \varphi[\mathcal{F}(\gamma, g)]$ for every $(\gamma, g) \in \Omega^u \overline{\text{Ell}}_M$.*

Proof. The proof is very similar to the proof of Lemma 7.9. We show first that $\Phi(\gamma, g)$ depends only on the isomorphism class of $\mathcal{F}(\gamma, g)$. Indeed, suppose that $\mathcal{F}(\gamma_1, g_1)$ is isomorphic (as a vector bundle) to $\mathcal{F}(\gamma_2, g_2)$ for $\gamma_i \in \Omega_{g_i} \overline{\text{Ell}}(E_i)$, $i = 1, 2$. By Lemma 7.13 and Property (P2^u), we can suppose without loss of generality that $\gamma_i \in \Omega_{g_i} \overline{\text{Dir}}(E_i)$. For sufficiently large integer k , the vector subbundles $\mathcal{F}(\gamma_1, g_1) \oplus o \oplus o$ and $o \oplus \mathcal{F}(\gamma_2, g_2) \oplus o$ of $\mathcal{E}_\partial^-(\gamma_1, g_1) \oplus \mathcal{E}_\partial^-(\gamma_2, g_2) \oplus k_{\partial M \times S^1}$ are homotopic. Then $\gamma_1 \oplus \gamma_2^+ \oplus \delta$ and $\gamma_1^+ \oplus \gamma_2 \oplus \delta$ are contained in the same path-connected component of $\Omega_{g_1 \oplus g_2 \oplus \text{Id}} \overline{\text{Dir}}(E_1 \oplus E_2 \oplus 2k_M)$, where γ_i^+ and δ are defined as in the proof of Lemma 7.9. By Properties (P1^u) and (P2^u), $\Phi(\gamma_1, g_1) + \Phi(\gamma_2^+, g_2) + \Phi(\delta, \text{Id}) = \Phi(\gamma_1^+, g_1) + \Phi(\gamma_2, g_2) + \Phi(\delta, \text{Id})$. Since $\mathcal{F}(\gamma_i^+, g_i) = o$, Lemma 7.14 implies $\Phi(\gamma_i^+, g_i) = o$. Therefore, $\Phi(\gamma_1, g_1) = \Phi(\gamma_2, g_2)$. The rest of the proof is the same as in Lemma 7.9. \square

Lemma 7.16. *The homomorphism φ vanishes on the image of $\pi^*: K^0(\partial M) \rightarrow K^0(\partial M \times S^1)$, where π denotes the projection of $\partial M \times S^1$ to ∂M .*

Proof. The proof is the same as the proof of Lemma 7.10. \square

Lemma 7.17. *The homomorphism φ vanishes on the image of $j^*: K^0(M \times S^1) \rightarrow K^0(\partial M \times S^1)$, where j denotes the embedding of $\partial M \times S^1$ to $M \times S^1$.*

Proof. It is sufficient to prove that $\varphi[j^*V] = o$ for any (isomorphism class of a) vector bundle V over $M \times S^1$. Let k be the rank of V . The lifting of V by the map $M \times [0, 1] \rightarrow M \times S^1$ is a trivial vector bundle $k_{M \times [0, 1]}$, so we can obtain V from this trivial bundle, gluing $k_{M \times \{1\}}$ with $k_{M \times \{0\}}$ by some unitary bundle automorphism $g \in U(k_M)$.

Let $E = k_M \oplus k_M$, $\tilde{g} = g \oplus g \in U(E)$, and let the symbol $\sigma \in \Sigma(E)$ be defined by formula (7.4). Since σ is \tilde{g}_* -invariant, the constant path $\beta: [0, 1] \rightarrow \overline{\Sigma}(E)$, $\beta(t) = (\sigma, -\text{Id})$, is an element of $\Omega_{\tilde{g}} \overline{\Sigma}(E)$. We can define a lifting $\gamma \in \Omega_{\tilde{g}} \overline{\text{Ell}}(E)$ of β by the formula $\gamma(t) = ((1-t)A_0 + tA_1, -\text{Id})$, where $A_0 = r(\sigma)$, $A_1 = \tilde{g}A_0\tilde{g}^{-1}$. Then $\mathcal{F}(\gamma, \tilde{g}) = \mathcal{E}_\partial^-(\gamma, \tilde{g})$ is isomorphic to $V|_{\partial M \times S^1} = j^*V$. By Lemma 7.15, $\Phi(\gamma, \tilde{g}) = \varphi[\mathcal{F}(\gamma, \tilde{g})] = \varphi[j^*V]$. By Lemma 7.14, $\Phi(\gamma, \tilde{g}) = o$. Thus $\varphi[j^*V] = o$. This completes the proof of the lemma. \square

Now we are ready to complete the proof of Theorem 7. By Lemma 7.15, there is a homomorphism $\varphi: K^0(\partial M \times S^1) \rightarrow \Lambda$ such that $\Phi(\gamma, g) = \varphi[\mathcal{F}(\gamma, g)]$ for every $(\gamma, g) \in \Omega^u \overline{\text{Ell}}_M$. By Lemmas 7.16 and 7.17, φ vanishes on the images of π^* and j^* . By Lemma 7.12, φ has the form $\varphi[\mathcal{F}] = c_1(\mathcal{F})[\partial M \times S^1] \cdot \lambda$ for some constant $\lambda \in \Lambda$. This completes the proof of the theorem. \square

8 Proof of Theorem 1

We deduce Theorem 1 from Theorem 7.

Lemma 8.1. *Let $D \in \text{Dir}(E)$, that is, D is a Dirac operator, odd with respect to chiral decomposition $E = E^+(D) \oplus E^-(D)$:*

$$(8.1) \quad D = \begin{pmatrix} 0 & D^- \\ D^+ & 0 \end{pmatrix}.$$

Let T be a positive definite automorphism of $E_\partial^-(D)$, and let L be the boundary condition for D defined by (5.1). Then D_T has no zero eigenvalues. The same is true for negative definite T . In particular, all operators from $\overline{\text{Dir}}_+(E)$, $\overline{\text{Dir}}_-(E)$, and $\overline{\text{Dir}}_\pm(E)$ have no zero eigenvalues.

Proof. Denote the symbol of D^+ by σ^+ . Let $u = (u^+, u^-)$ be a section of the vector bundle $E = E^+(D) \oplus E^-(D)$ belonging to the kernel of D_L . Then for the restriction of u to ∂M we have $i\sigma^+(n)u^+ = Tu^-$. By Green's formula,

$$\int_{\partial M} \langle Tu^-, u^- \rangle dl = \int_{\partial M} \langle i\sigma^+(n)u^+, u^- \rangle dl = \int_M (\langle D^+u^+, u^- \rangle - \langle u^+, D^-u^- \rangle) ds = 0,$$

where dl is the length element on ∂M and ds is the volume element on M .

Suppose now that T is positive definite on ∂M . Then $u^- \equiv 0$ on ∂M . This together with the boundary condition implies that $u^+ = (i\sigma^+(n))^{-1}u^- \equiv 0$ on ∂M , so the restriction of u to ∂M vanishes. By the weak inner unique continuation property of Dirac operators [2], we get $u \equiv 0$ on whole M . So in this case D_L has no zero eigenvalues. The same reasoning shows that D_L has no zero eigenvalues if T is negative definite on ∂M . This completes the proof of the lemma. \square

The spectral flow defines an integer-valued function on $\Omega^u \overline{\text{Ell}}_M$ satisfying properties $(\text{So}^u - \text{S}_2^u)$. By Lemma 8.1, (So^u) implies $(\widetilde{\text{Po}}^u)$. Thus the spectral flow satisfies conditions of Theorem 7, with $\Phi = \text{sf}$ and $\Lambda = \mathbb{Z}$. By Theorem 7, there is an integer λ such that

$$(8.2) \quad \text{sf}(\gamma) = c_1(\mathcal{F}(\gamma, g))[\partial M \times S^1] \cdot \lambda$$

for every trivial vector bundle E over M , every $g \in \text{U}(E)$, and every $\gamma \in \Omega_g \overline{\text{Ell}}(E)$. Every vector bundle over M is trivial, so it remains only to check that $\lambda = 1$.

Lemma 8.2. *The value of λ does not depend on the choice of a metric on M .*

Proof. Let h, h' be two metrics on M . The Hilbert spaces $L^2(M, h; E)$ and $L^2(M, h'; E)$ are isomorphic, with an isometry given by the formula $u \mapsto cu$, where c is the positive-valued function on M defined by the formula $c = \sqrt{\det(h')/\det(h)}$. This isometry induces the bijection between the spaces $\overline{\text{Ell}}(M, h; E)$ and $\overline{\text{Ell}}(M, h'; E)$ and leaves invariant the spectral flow of paths. On the other hand, such isometry leaves invariant both symbols of operators and local boundary conditions, so it leaves invariant $F(A, L)$. The conjugation by c leaves invariant bundle automorphism g as well. Therefore, the aforementioned bijection $\overline{\text{Ell}}(M, h; E) \rightarrow \overline{\text{Ell}}(M, h'; E)$ does not affect $\mathcal{F}(\gamma, g)$. This implies that

the factor λ in (8.2) is the same for metrics h and h' . Since h and h' are arbitrary metrics, λ does not depend on the choice of the metric. \square

Therefore, we can choose any convenient metric h on M and any convenient triple (E, g, γ) for which the right-hand side of (5.6) does not vanish; then the computation of the spectral flow gives us the value of $\lambda = \lambda_M$ for a given surface M .

Lemma 8.3. *If M is diffeomorphic to the annulus, then $\lambda_M = \lambda_{\text{ann}} = 1$.*

Proof. This was proven by the author in [15, Theorem 4] (λ_{ann} is denoted by c_2 there). The proof is based on the direct computation of the spectral flow for the Dirac operator on $S^1 \times [0, 1]$ with varying connection and fixed boundary condition. \square

The following lemma, when combined with Lemma 8.3, implies the identity $\lambda_M = 1$ for an arbitrary compact connected oriented surface M and thus completes the proof of Theorem 1.

Lemma 8.4. *For any smooth oriented connected surface M the values of λ_M and λ_{ann} coincide.*

Proof. There are different ways to reduce the computation of λ_M to the case of an annulus. Here we describe one of them, namely the splitting of M into two pieces: the smaller surface M' diffeomorphic to M and the collar M'' of the boundary. Following the ideas of P. Kirk and M. Lesch from [12], we take the Dirac operator which has the product form near boundary and choose mutually orthogonal boundary conditions on the sides of the cut. Then the spectral flow over M coincides with the sum of spectral flows over M' and M'' . Since M'' is the disjoint union of annuli, this reasoning allows to reduce the computation of λ_M to the computation for the annuli. Let us describe this procedure in more detail.

Let U be a collar neighbourhood of ∂M in M ; we identify U with the product $(-2\varepsilon, 0] \times \partial M$. Let (y, z) be the coordinates on U , with $y \in \partial M$, $z \in (-2\varepsilon, 0]$, and (∂_z, ∂_y) a positive oriented basis in TU . Equip M with a metric whose restriction to U has the product form $dl^2 = dy^2 + dz^2$.

Let D be a Dirac operator acting on sections of $E = 2_M \oplus 2_M$ such that its restriction to U has the form $D|_U = -i(\sigma_1 \partial_z + \sigma_2 \partial_y)$, where $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$. To obtain such an operator D , we can take arbitrary lifting D' of σ to $\text{Dir}(E)$. Its restriction to U has the form $D' = \sigma_1 \partial_z + \sigma_2 \partial_y + C$ for some self-adjoint bundle endomorphism C of $E|_U$. Smoothly continuing C to the rest of M and subtracting it from D' , we obtain the desired operator D .

Let \mathcal{F} be a vector bundle of rank 1 over $\partial M \times S^1$ such that $c_1(\mathcal{F})[\partial M \times S^1] \neq 0$. Choose the smooth embedding of \mathcal{F} into the trivial vector bundle of rank 2 over $\partial M \times S^1$. Restricting this embedding to $\partial M \times \{t\}$, we obtain the smooth loop $(F_t)_{t \in S^1}$ of smooth subbundles F_t of $2_{\partial M}$. Define the smooth automorphisms T_t of $2_{\partial M}$ by the formula $T_t = (-1)_{F_t} \oplus 1_{F_t^\perp}$.

Let $L_t \subset E_\partial$ be the correspondent boundary condition for D (that is, L_t is obtained from T_t as described in Proposition 5.1). Then $\mathcal{F} = \mathcal{F}(\gamma)$ for the loop $\gamma \in \Omega \overline{\text{Ell}}(E)$ defined by the formula $\gamma(t) = (D, L_t)$. By Theorem 6,

$$\text{sf}(D, L_t) = c_1(\mathcal{F})[\partial M \times S^1] \cdot \lambda_M.$$

Let us cut M along $N = \{-\varepsilon\} \times \partial M \subset U$. We obtain the disconnected surface $M^{\text{cut}} = M' \amalg M''$, where $M'' = [-\varepsilon, 0] \times \partial M$ is the disjoint union of annuli and M' is diffeomorphic to M . Denote by $E^{\text{cut}} = E' \amalg E''$ the lifting of E on M^{cut} , and by $D^{\text{cut}} = D' \amalg D''$ the lifting of D on M^{cut} . By N', N'' denote the sides of the cut, so that $\partial M' = N'$ and $\partial M'' = N'' \amalg \partial M$.

The restriction of E^{cut} to $N' \amalg N''$ is isomorphic to the disjoint union of two copies of $E|_N$. Let us identify its sections with sections of the vector bundle $\bar{E}_\partial = (E \oplus E)|_N$. The diagonal subbundle $\Delta = \{u \oplus u\}$ of \bar{E}_∂ defines the so called transmission boundary condition on the cut. The natural isometry $L^2(M; E) \rightarrow L^2(M^{\text{cut}}; E^{\text{cut}})$ takes the operator D_{L_t} to the operator $D_{\Delta \amalg L_t}^{\text{cut}}$. Therefore, $D_{\Delta \amalg L_t}^{\text{cut}}$ is self-adjoint Fredholm regular operator on $L^2(M^{\text{cut}}; E^{\text{cut}})$, and

$$\text{sf}(D, L_t) = \text{sf}(D^{\text{cut}}, \Delta \amalg L_t).$$

Extending the identification above to the identification of sections of $E^{\text{cut}}|_{U' \amalg U''}$ with sections of $\bar{E} = (E \oplus E)|_{U'}$, where $U' = (-2\varepsilon, -\varepsilon] \times \partial M$, $U'' = [-\varepsilon, 0] \times \partial M$, we can write D^{cut} in the collar of the cut as

$$\bar{D} = -i(\bar{\sigma}_1 \partial_{\bar{z}} + \bar{\sigma}_2 \partial_y), \text{ where } \bar{\sigma}_1 = \begin{pmatrix} \sigma_1 & 0 \\ 0 & -\sigma_1 \end{pmatrix}, \bar{\sigma}_2 = \begin{pmatrix} \sigma_2 & 0 \\ 0 & \sigma_2 \end{pmatrix},$$

and \bar{z} is the normal coordinate increasing in the direction of the cut (so $\bar{z} = z$ on U' and $\bar{z} = -z - 2\varepsilon$ on U''). We also change the orientation on M' , so that $(\partial_{\bar{z}}, \partial_y)$ becomes a negative oriented basis. Then

$$(8.3) \quad \bar{E}^+ = E'^- \oplus E''^+, \quad \bar{E}^- = E'^+ \oplus E''^-.$$

Denote by $\bar{\sigma}_1^+$ the restriction of $\bar{\sigma}_1$ to \bar{E}^+ . Proposition 5.1 associates with every self-adjoint automorphism \bar{T} of \bar{E}_∂^- the subbundle $\bar{L}(\bar{T})$ of \bar{E}_∂ given by the formula $i\bar{\sigma}_1^+ \bar{u}^+ = \bar{T} \bar{u}^-$. Each $\bar{L}(\bar{T})$ is a self-adjoint well posed boundary condition for \bar{D} on the cut, so $\bar{L}(\bar{T}) \amalg L_t$ is a self-adjoint well posed boundary condition for D^{cut} .

We have $\bar{\sigma}_1^+ = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ with respect to decompositions (8.3), so the transmission boundary condition Δ corresponds to the automorphism $\bar{T}_\Delta = i\bar{\sigma}_1^+ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ of \bar{E}_∂^- . Let (\bar{T}_s) be a smooth homotopy connecting $\bar{T}_0 = \bar{T}_\Delta$ with $\bar{T}_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ in the space of self-adjoint bundle automorphisms of \bar{E}_∂^- . Denote by \bar{L}_s the subbundle of \bar{E}_∂ corresponding to \bar{T}_s , and let $\bar{L} = \bar{L}_1$. Then $\bar{L}_s \amalg L_t$ is a self-adjoint well posed global boundary condition for D^{cut} , so $D_{\bar{L}_s \amalg L_t}^{\text{cut}}$ is a regular self-adjoint Fredholm operator on $L^2(M^{\text{cut}}; E^{\text{cut}})$ for each s, t . By Proposition A.9 from the Appendix, the map

$$[0, 1] \times S^1 \rightarrow \text{Gr}\left(H^{1/2}(N; \bar{E}) \oplus H^{1/2}(\partial M; E_\partial)\right) \cong \text{Gr}\left(H^{1/2}(\partial M^{\text{cut}}; E_\partial^{\text{cut}})\right),$$

$(s, t) \mapsto H^{1/2}(\bar{L}_s) \oplus H^{1/2}(L_t)$, is continuous. By Propositions A.7 and A.8, this implies the continuity of the map

$$[0, 1] \times S^1 \rightarrow \mathcal{RF}^{\text{sa}}(L^2(M^{\text{cut}}; E^{\text{cut}})), \quad (s, t) \mapsto D_{\bar{L}_s \amalg L_t}^{\text{cut}}.$$

Therefore, by the homotopy invariance property of the spectral flow we have

$$\text{sf}(D^{\text{cut}}, \Delta \amalg L_t) = \text{sf}(D^{\text{cut}}, \bar{L} \amalg L_t).$$

The boundary condition \bar{L} is given by the formula $i\bar{\sigma}_1^+ \bar{u}^+ = \bar{T}_1 \bar{u}^-$, that is, $i\bar{u}^+ = \bar{u}^-$. Coming back from \bar{E}_∂ to $E^{\text{cut}}|_{N' \amalg N''}$, we obtain $L' \amalg L''$ in place of \bar{L} , where L' is the subbundle of $E^{\text{cut}}|_{N'}$ given by the formula $i u'^- = u'^+$ and L'' is the subbundle of $E^{\text{cut}}|_{N''}$ given by the formula $i u''^+ = u''^-$. Therefore, $\bar{L} \amalg L_t$ is a *local* boundary condition for D^{cut} . Applying Theorem 6 to the connected components of M^{cut} , we obtain

$$\begin{aligned} \text{sf}(D^{\text{cut}}, \bar{L} \amalg L_t) &= \text{sf}(D', L') + \text{sf}(D'', L'' \amalg L_t) = \text{sf}(D'', L'' \amalg L_t) = \\ &= (c_1(\mathcal{F}'')[N'' \times S^1] + c_1(\mathcal{F})[\partial M \times S^1]) \cdot \lambda_{\text{ann}} = c_1(\mathcal{F})[\partial M \times S^1] \cdot \lambda_{\text{ann}}, \end{aligned}$$

since \mathcal{F}'' is zero vector bundle.

Combining all this together, we obtain

$$c_1(\mathcal{F})[\partial M \times S^1] \cdot \lambda_M = \text{sf}(D, L_t) = \text{sf}(D^{\text{cut}}, L' \amalg L'' \amalg L_t) = c_1(\mathcal{F})[\partial M \times S^1] \cdot \lambda_{\text{ann}}.$$

The value of $c_1(\mathcal{F})[\partial M \times S^1]$ does not vanish due to the choice of \mathcal{F} . Therefore, $\lambda_{\text{ann}} = \lambda_M$, which completes the proof of the lemma and of Theorem 1. \square

9 Proofs of Theorems 2–5

Proof of Theorem 4. Let $\Phi: \Omega \overline{\text{Ell}}_M \rightarrow \Lambda$ satisfies properties $(\widetilde{P}_0, P_1, P_2)$. By Theorem 6, there is a constant $\lambda \in \Lambda$ such that $\Phi(\gamma) = c_1(\mathcal{F}(\gamma))[\partial M \times S^1] \cdot \lambda$ for all $\gamma \in \Omega \overline{\text{Ell}}_M$. Substituting expression (1.4) to this formula, we obtain

$$(9.1) \quad \Phi(\gamma) = \text{sf}(\gamma) \cdot \lambda.$$

Conversely, let Φ be given by formula (9.1). The spectral flow satisfies properties $(\widetilde{P}_0, P_1, P_2)$, so Φ satisfies them too. This completes the proof of the Theorem. \square

Proof of Theorem 2. Let $\Phi: \Omega \overline{\text{Ell}}_M \rightarrow \Lambda$ satisfies properties (P_0-P_2) . Every constant loop in $\overline{\text{Ell}}(E)$ is homotopic to a constant loop of invertible operators, so Φ vanishes on constant loops. By Lemma 8.1, Φ vanishes on loops in $\overline{\text{Dir}}_\pm(2k_M)$. Therefore, Φ satisfies conditions of Theorem 6. By Theorem 6, there is a constant $\lambda \in \Lambda$ such that

$\Phi(\gamma) = c_1(\mathcal{F}(\gamma))[\partial M \times S^1] \cdot \lambda$ for all $\gamma \in \Omega \overline{\text{Ell}}_M$. Substituting expression (1.4) to this formula, we obtain (9.1).

Conversely, let Φ be given by formula (9.1). The spectral flow satisfies properties (Po–P2), so Φ satisfies them too. This completes the proof of the theorem. \square

Proof of Theorem 5. Let $\Phi: \Omega^u \overline{\text{Ell}}_M \rightarrow \Lambda$ satisfies properties $(\widetilde{\text{Po}}^u, \text{P1}^u, \text{P2}^u)$. By Theorem 7, there is a constant $\lambda \in \Lambda$ such that $\Phi(\gamma, g) = c_1(\mathcal{F}(\gamma, g))[\partial M \times S^1] \cdot \lambda$ for all $(\gamma, g) \in \Omega^u \overline{\text{Ell}}_M$. Substituting expression (1.4) to this formula, we obtain

$$(9.2) \quad \Phi(\gamma, g) = \text{sf}(\gamma) \cdot \lambda.$$

Conversely, let Φ be given by formula (9.2). The spectral flow satisfies properties $(\widetilde{\text{Po}}^u, \text{P1}^u, \text{P2}^u)$, so Φ satisfies them too. This completes the proof of the theorem. \square

Proof of Theorem 3. Let $\Phi: \Omega^u \overline{\text{Ell}}_M \rightarrow \Lambda$ satisfies properties $(\text{Po}^u, \text{P2}^u)$. Every constant loop in $\overline{\text{Ell}}(E)$ is homotopic to a constant loop of invertible operators, so Φ vanishes on constant loops. By Lemma 8.1, Φ vanishes on loops in $\overline{\text{Dir}}_{\pm}(2k_M)$. Therefore, Φ satisfies conditions of Theorem 7. By Theorem 7, there is a constant $\lambda \in \Lambda$ such that $\Phi(\gamma, g) = c_1(\mathcal{F}(\gamma, g))[\partial M \times S^1] \cdot \lambda$ for all $(\gamma, g) \in \Omega^u \overline{\text{Ell}}_M$. Substituting expression (1.4) to this formula, we obtain (9.2).

Conversely, let Φ be given by formula (9.2). The spectral flow satisfies properties $(\text{Po}^u, \text{P2}^u)$, so Φ satisfies them too. This completes the proof of the theorem. \square

A Appendix. Gap continuity of closed operators

In this Appendix we give some general conditions describing if a family of closed operators (in particular, differential operators on a manifold with boundary) is gap continuous. We use these results, namely Propositions A.7 and A.9, in the main part of the paper for two purposes: first, to prove Proposition 4.1 (see the proof in the end of the Appendix); second, to show the continuity of the family of global boundary value problems used in the proof of Lemma 8.4.

After the main part of the Appendix (namely, the case of Hilbert spaces) was written, the author discovered that some of these results, though in a different form and with different proofs, are contained in the Appendix to the recent paper of Booss-Bavnbek and Zhu [3]. In particular, our Proposition A.4 is a corollary of [3, Proposition A.6.2]. Nevertheless, we leave these results and their proofs in the paper for the sake of completeness, and also because their statements better meet our needs. For Hilbert spaces, our proofs have the advantage of not using elaborated estimates and inequalities. We also add more general case of Banach spaces to the Appendix with the purpose of better

matching the results of [3], though we use only the Hilbert spaces case in the main part of the paper.

It is worth noticing that our Proposition A.3 gives the equivalent definition of the gap topology on the space $\text{Gr}(H)$ of all complemented closed linear subspaces of a Banach space H . Namely, the gap topology on $\text{Gr}(H)$ coincides with the quotient topology induced by the map $\text{Proj}(H) \rightarrow \text{Gr}(H)$, $P \mapsto \text{Im } P$, where $\text{Proj}(H)$ is the space of all idempotents in $\mathcal{B}(H)$ with the norm topology. The author does not know if this fact was noted before.

Closed subspaces

Let H be a Banach space. Denote by $\mathcal{B}(H)$ the space of all bounded linear operators on H with the norm topology. Denote by $\text{Proj}(H)$ the subspace of $\mathcal{B}(H)$ consisting of all idempotents.

A closed subspace $L \subset H$ is called complemented if there is another closed subspace $M \subset H$ such that $L \cap M = 0$, $L + M = H$; such pair is called a complementary pair. Equivalently, $L \subset H$ is called complemented if it is the image of some $P \in \text{Proj}(H)$; (L, M) is called a complementary pair if $(L, M) = (\text{Im } P, \text{Ker } P)$ for some $P \in \text{Proj}(H)$. We denote by $\text{Gr}(H)$ the set of all complemented closed linear subspaces of H , and by $\text{Gr}^{(2)}(H)$ the set of all complementary pairs of subspaces of H . We will also write $\text{Gr}^2(H)$ instead of $\text{Gr}(H)^2$ for convenience.

For $(L, M) \in \text{Gr}^{(2)}(H)$ we denote by $P_{L,M}$ the projection of H onto L along M . For $M \in \text{Gr}(H)$ denote by $\text{Gr}^M(H) = \{L \in \text{Gr}(H) : (L, M) \in \text{Gr}^{(2)}(H)\}$ the space of all complement subspaces for M .

The gap topology on $\text{Gr}(H)$ is defined by the metric

$$\hat{\delta}(L_1, L_2) = \max_{i \neq j} \left\{ \sup \left\{ \text{dist}(u, L_j) : u \in L_i, \|u\| = 1 \right\} \right\}, \quad \hat{\delta}(0, 0) = 0, \quad \hat{\delta}(0, L) = 1 \text{ for } L \neq 0.$$

$\text{Gr}^{(2)}(H)$ is topologized as a subset of $\text{Gr}(H)^2$.

If H is a Hilbert space, then each closed linear subspace of H is complemented, so $\text{Gr}(H)$ is the space of all closed linear subspaces of H . The gap topology on $\text{Gr}(H)$ coincides with the topology induced by the gap metric $\delta(L, M) = \|P_L - P_M\|$, where P_L denotes the orthogonal projection of H onto L .

Proposition A.1. *Let H be a Banach space and $P, Q \in \text{Proj}(H)$. Then the following two conditions are equivalent:*

1. *Both $(\text{Im } P, \text{Im } Q)$ and $(\text{Ker } P, \text{Ker } Q)$ lie in $\text{Gr}^{(2)}(H)$.*

2. $P - Q$ is invertible.

If this is the case, then for the projection S on $\text{Im } P$ along $\text{Im } Q$ and the projection T on $\text{Ker } P$ along $\text{Ker } Q$ we have:

$$(A.1) \quad S = P(P - Q)^{-1}, \quad T = (P - 1)(P - Q)^{-1}, \quad (P - Q)^{-1} = S - T,$$

and $P + Q = (2S - 1)(P - Q)$ is also invertible.

Proof. ($1 \Rightarrow 2$) Let $(\text{Im } P, \text{Im } Q), (\text{Ker } P, \text{Ker } Q) \in \text{Gr}^{(2)}(H)$. Denote by S, T the elements of $\text{Proj}(H)$ corresponding these two pairs of complementary subspaces. Using the identities $SP = P, TQ = T, SQ = 0$, and $(1 - T)(1 - P) = 0$, we obtain

$$(S - T)(P - Q) = T + P - TP = 1 - (1 - T)(1 - P) = 1.$$

Similarly, we have

$$(P - Q)(S - T) = Q + S - QS = 1 - (1 - Q)(1 - S) = 1.$$

Therefore, $P - Q$ is invertible with $S - T$ the inverse operator.

($2 \Rightarrow 1$) Let $P - Q$ be invertible. It vanishes on the intersections $\text{Im } P \cap \text{Im } Q$ and $\text{Ker } P \cap \text{Ker } Q$, so these intersections are trivial. Consider the operators $S = P(P - Q)^{-1}$ and $S' = -Q(P - Q)^{-1}$. We have $\text{Im } S = \text{Im } P$, $\text{Im } S' = \text{Im } Q$, and $S + S' = 1$, so $\text{Im } P + \text{Im } Q = H$. Similarly, consider the operators $T = (P - 1)(P - Q)^{-1}$ and $T' = (1 - Q)(P - Q)^{-1}$. We have $\text{Im } T = \text{Ker } P$, $\text{Im } T' = \text{Ker } Q$, and $T + T' = 1$, so $\text{Ker } P + \text{Ker } Q = H$. All four subspaces $\text{Im } P, \text{Im } Q, \text{Ker } P, \text{Ker } Q$ are closed. Therefore, both $(\text{Im } P, \text{Im } Q)$ and $(\text{Ker } P, \text{Ker } Q)$ lie in $\text{Gr}^{(2)}(H)$. \square

Corollary A.1. Let H be a Hilbert space. Then the following statements hold:

1. The pair (L, M) of closed subspaces of H is complementary if and only if $P_L - P_M$ is invertible. If this is the case, then

$$(A.2) \quad P_{L,M} = P_L(P_L - P_M)^{-1}.$$

2. Let $P \in \text{Proj}(H)$. Then the operator $P + P^* - 1$ is invertible, and the orthogonal projection on the image of P is given by the formula

$$(A.3) \quad P^{\text{ort}} = P(P + P^* - 1)^{-1}.$$

Proof. 1. If $(L, M) \in \text{Gr}^{(2)}(H)$, then $(L^\perp, M^\perp) \in \text{Gr}^{(2)}(H)$ too. Applying Proposition A.1 to the pair of orthogonal projections P_L and P_M , we obtain the first claim of the Corollary.

2. $1 - P^*$ is the projection on $(\text{Im } P)^\perp$ along $(\text{Ker } P)^\perp$. Applying Proposition A.1 to the pair of projections P and $1 - P^*$, we see that $P + P^* - 1 = P - (1 - P^*)$ is invertible and $P(P + P^* - 1)^{-1}$ is the projection on $\text{Im } P$ along $(\text{Im } P)^\perp$. \square

Proposition A.2. *Let H be a Banach space. Then the map $\text{Im}: \text{Proj}(H) \rightarrow \text{Gr}(H)$ is continuous, the map $\varphi: \text{Proj}(H) \rightarrow \text{Gr}^{(2)}(H)$ taking $P \in \text{Proj}(H)$ to $(\text{Im } P, \text{Ker } P) \in \text{Gr}^{(2)}(H)$ is a homeomorphism, and $\text{Gr}^{(2)}(H)$ is open in $\text{Gr}^2(H)$.*

We first give the proof in the case of a Hilbert space H , because it is simpler and because we need only this case in the main part of the paper as well as in the proofs of all the results below in the context of Hilbert spaces. After proving the “Hilbert case” we give the proof of the general “Banach case”.

Proof. Suppose first that H is a Hilbert space.

The map $\text{Im}: \text{Proj}(H) \rightarrow \text{Gr}(H)$ is continuous. Indeed, it is the composition of the two maps $\text{Proj}(H) \rightarrow \text{Proj}^{\text{ort}}(H)$ and $\text{Im}: \text{Proj}^{\text{ort}}(H) \rightarrow \text{Gr}(H)$, where the first map is given by formula (A.3) and $\text{Proj}^{\text{ort}}(H)$ is the subspace of $\text{Proj}(H)$ consisting of orthogonal projections. The first map is continuous and the second map is an isometry, so their composition is also continuous.

The conjugation by the involution $P \mapsto 1 - P$ takes the map $\text{Im}: \text{Proj}(H) \rightarrow \text{Gr}(H)$ to the map $\text{Ker}: \text{Proj}(H) \rightarrow \text{Gr}(H)$, so the second map is also continuous. Therefore, φ is continuous. Obviously, φ is bijective.

The inverse map $\text{Gr}^{(2)}(H) \rightarrow \text{Proj}(H)$ is given by formula (A.2) and therefore is continuous. Thus the map $\text{Proj}(H) \rightarrow \text{Gr}^{(2)}(H)$ is a homeomorphism.

To prove that $\text{Gr}^{(2)}(H)$ is open in $\text{Gr}^2(H)$, take arbitrary $(L, M) \in \text{Gr}^{(2)}(H)$. The operator $P_L - P_M$ is invertible by Corollary A.1. Choose $\varepsilon > 0$ such that 2ε -neighbourhood of $P_L - P_M$ in $\mathcal{B}(H)$ consists of invertible operators. Then for any $L', M' \in \text{Gr}(H)$ such that $\|P_L - P_{L'}\| < \varepsilon$, $\|P_M - P_{M'}\| < \varepsilon$ we have

$$\|(P_L - P_M) - (P_{L'} - P_{M'})\| \leq \|P_L - P_{L'}\| + \|P_M - P_{M'}\| < 2\varepsilon,$$

so $P_{L'} - P_{M'}$ is invertible. Applying again Corollary A.1, we obtain $(L', M') \in \text{Gr}^{(2)}(H)$. This completes the proof of the proposition for Hilbert spaces.

Let now H be an arbitrary Banach space.

The continuity of the map $\text{Im}: \text{Proj}(H) \rightarrow \text{Gr}(H)$ follows from the inequality $\hat{\delta}(\text{Im } P, \text{Im } Q) \leq \|P - Q\|$. As above, this implies that φ is a continuous bijection. The continuity of the map $\text{Gr}^{(2)}(H) \rightarrow \text{Proj}(H)$, $(L, M) \mapsto P_{L, M}$ follows from [10, Lemma 0.2]. By [5, Lemma 1 and Theorem 2], $\text{Gr}^{(2)}(H)$ is open in $\text{Gr}^2(H)$. This completes the proof of the Proposition for Banach spaces. \square

Proposition A.3. *Let H be a Banach space. Then the gap topology on $\text{Gr}(H)$ coincides with the quotient topology induced by the map $\text{Im}: \text{Proj}(H) \rightarrow \text{Gr}(H)$, $P \mapsto \text{Im } P$.*

Proof. The projection $p_1: \text{Gr}^2(H) \rightarrow \text{Gr}(H)$ onto the first factor is an open continuous map. By Proposition A.2, $\text{Gr}^{(2)}(H)$ is open in $\text{Gr}^2(H)$, so the restriction of p_1 to $\text{Gr}^{(2)}(H)$

is also an open map. This restriction maps $\text{Gr}^{(2)}(H)$ continuously and surjectively onto $\text{Gr}(H)$. Therefore, the gap topology on $\text{Gr}(H)$ coincides with the quotient topology induced by the map $p_1: \text{Gr}^{(2)}(H) \rightarrow \text{Gr}(H)$. To complete the proof, it is sufficient to apply the homeomorphism $\varphi: \text{Proj}(H) \rightarrow \text{Gr}^{(2)}(H)$ from Proposition A.2. \square

Proposition A.4. *Let $j \in \mathcal{B}(H, H')$ be an injective map of Banach spaces. Denote by $\text{Gr}_j(H)$ the subspace of $\text{Gr}(H)$ consisting of L with $j(L) \in \text{Gr}(H')$. Then $\text{Gr}_j(H)$ is open in $\text{Gr}(H)$ and the natural inclusion $j_*: \text{Gr}_j(H) \hookrightarrow \text{Gr}(H')$, $L \mapsto j(L)$ is continuous.*

Proof. By Proposition A.2, $\text{Gr}^M(H)$ is open in $\text{Gr}(H)$. Thus the statement of the proposition results from the following lemma.

Lemma A.1. *Let $L \in \text{Gr}_j(H)$, let $M' \in \text{Gr}(H')$ be a complement subspace for $L' = j(L)$, and $M = j^{-1}(M')$. Then $L \in \text{Gr}^M(H) \subset \text{Gr}_j(H)$, and the restriction of j_* to $\text{Gr}^M(H)$ is continuous.*

Proof of the Lemma. Denote by Q' the projection of H' onto L' along M' . By the Closed Graph Theorem, the bounded linear operator $j|_L: L \rightarrow L'$ is an isomorphism. Thus the composition $Q = (j|_L)^{-1}Q'j$ is a bounded operator on H . Obviously, Q is an idempotent, $\text{Im } Q = L$, and $\text{ker } Q = M$. This implies that L and M are complement subspaces of H .

Let $N \in \text{Gr}^M(H)$, $N' = j(N)$. Then $Q_N = jP_{N,M}(j|_L)^{-1}Q'$ is a bounded operator acting on H' . $\text{Ker } Q_N = M'$ and the restriction of $Q_N^2 - Q_N$ to L' vanishes, so $Q_N^2 = Q_N$ and $Q_N \in \text{Proj}(H')$. $\text{Im } Q_N \subset N'$ and $N' \cap M' = j(N \cap M) = 0$. Therefore, $Q_N = P_{N',M'}$, $N' = \text{Im } Q_N \in \text{Gr}(H')$, and $N \in \text{Gr}_j(H)$.

By Proposition A.2, the map $N \mapsto P_{N,M}$ is continuous. Thus the map $\text{Gr}^M(H) \rightarrow \text{Proj}(H')$, $N \mapsto Q_N$ is also continuous. Composing it with the continuous map $\text{Im}: \text{Proj}(H') \rightarrow \text{Gr}(H')$, we obtain the continuity of the map $j_*: \text{Gr}^M(H) \rightarrow \text{Gr}(H')$, $N \mapsto j(N) = \text{Im } Q_N$. This completes the proof of the lemma and of Proposition A.4. \square

Closed operators

Let H, H' be Hilbert spaces. The space $\mathcal{C}(H, H')$ of closed linear operators from H to H' is the subspace of $\text{Gr}(H \oplus H')$ consisting of closed subspaces of $H \oplus H'$ which injectively projects on H . An element of $\mathcal{C}(H, H')$ can be identified with a linear (not necessarily bounded) operator A acting to H' from (not necessarily closed or dense) subspace $\text{dom}(A)$ of H such that the graph of A is a closed subspace of $H \oplus H'$.

Results of this subsection remain valid for Banach spaces as well. However, in this case the space $\mathcal{C}(H, H')$ as we defined it (namely, as a the subspace of $\text{Gr}(H \oplus H')$) does not contain all closed linear operators from H to H' , but only those whose graphs are complemented subspaces of $H \oplus H'$. Nevertheless, the families of such operators often arise in applications, so these results can be used for them as well. In particular, for Banach spaces H, H' and a linear operator A acting from $\mathcal{D} \subset H$ to H' , if $\text{Ker } A \subset H$

and $\text{Im } A \subset H'$ are closed complemented subspaces, then the graph of A is a closed complemented subspace of $H \oplus H'$. In particular, each (unbounded) Fredholm operator has this property.

Proposition A.5. *Let H, H' be Banach spaces. Then the map $\mathcal{B}(H, H') \times \text{Gr}(H) \rightarrow \mathcal{C}(H, H')$, $(A, \mathcal{D}) \mapsto A|_{\mathcal{D}}$ is continuous.*

Proof. For each $A \in \mathcal{B}(H, H')$ we define the automorphism J_A of $H \oplus H'$ by the formula $J_A(u \oplus u') = u \oplus (u' - Au)$. Both $A \mapsto J_A$ and $A \mapsto J_A^{-1}$ are continuous maps from $\mathcal{B}(H, H')$ to $\mathcal{B}(H \oplus H')$. The formula $f(A, Q) = J_A^{-1} Q P_{H, H'} J_A$ defines the continuous map $f: \mathcal{B}(H, H') \times \text{Proj}(H) \rightarrow \text{Proj}(H \oplus H')$ (here $P_{H, H'}$ denotes the projection of $H \oplus H'$ on H along H'). Since J_A takes the graph of $A|_{\mathcal{D}}$ to $\mathcal{D} \oplus 0$ for each $\mathcal{D} \in \text{Gr}(H)$, $f(A, Q)$ is the projection of $H \oplus H'$ onto the graph of $A|_{\text{Im } Q}$. In other words, we have the commutative diagram

$$\begin{array}{ccc} \mathcal{B}(H, H') \times \text{Proj}(H) & \xrightarrow{f} & \text{Proj}(H \oplus H') \\ \downarrow \text{Id} \times \text{Im} & & \downarrow \text{Im} \\ \mathcal{B}(H, H') \times \text{Gr}(H) & \xrightarrow{g} & \text{Gr}(H \oplus H') \end{array}$$

where g is the map taking the pair (A, \mathcal{D}) to the graph of $A|_{\mathcal{D}}$. The top and the right arrows of the diagram are continuous maps, while the left arrow is a quotient map by Proposition A.3. Therefore, g is also continuous. This completes the proof of the proposition. \square

Proposition A.6. *Let W, H, H' be Banach spaces, and let $j \in \mathcal{B}(W, H)$ be injective. Denote by $\mathcal{C}_j(W, H')$ the subspace of $\mathcal{C}(W, H')$ consisting of operators $A: \text{dom}(A) \rightarrow H'$ such that the operator $j_* A: j(\text{dom}(A)) \rightarrow H'$, $j_* A = A \cdot j^{-1}$ lies in $\mathcal{C}(H, H')$. Then the natural inclusion $j_*: \mathcal{C}_j(W, H') \hookrightarrow \mathcal{C}(H, H')$ is continuous.*

Proof. Consider the following commutative diagram:

$$\begin{array}{ccccc} \mathcal{C}(W, H') & \hookleftarrow & \mathcal{C}_j(W, H') & \xrightarrow{j_*} & \mathcal{C}(H, H') \\ \downarrow & & \downarrow & & \downarrow \\ \text{Gr}(W \oplus H') & \hookleftarrow & \text{Gr}_j(W \oplus H') & \xrightarrow{j_*} & \text{Gr}(H \oplus H') \end{array}$$

The spaces above are just subspaces on the spaces below, and $\mathcal{C}_j(W, H') = \mathcal{C}(W, H') \cap \text{Gr}_j(W \oplus H')$. By Proposition A.4, the map $j_*: \text{Gr}_j(W \oplus H') \rightarrow \text{Gr}(H \oplus H')$ is continuous. So the restriction of j_* to $\mathcal{C}_j(W, H') \subset \text{Gr}_j(W \oplus H')$ is also continuous. This completes the proof of the proposition. \square

Differential and pseudo-differential operators

The results of the previous subsection can be used for d -th order differential and pseudo-differential operators acting between sections of vector bundles over M . To achieve

continuity of the corresponding families of closed operators, the relevant topology on the space of d -th order differential operators should be the C^0 -topology on coefficients.

Indeed, let E, E' be smooth Hermitian vector bundles over a smooth manifold M . For an integer $d \geq 1$, we denote by $\text{Op}^d(E, E')$ the set of all pairs (A, \mathcal{D}) such that A is a bounded operator from $H^d(M; E)$ to $L^2(M; E')$, \mathcal{D} is a closed subspace of $H^d(M; E)$, and the restriction $A|_{\mathcal{D}}$ of A to the domain \mathcal{D} is closed as an operator from $L^2(M; E)$ to $L^2(M; E')$. Here $H^d(M; E)$ denotes the d -th order Sobolev space of sections of E . Let us equip $\text{Op}^d(E, E')$ with the topology induced by the inclusion

$$\text{Op}^d(E, E') \hookrightarrow \mathcal{B}(H^d(M; E), L^2(M; E')) \times \text{Gr}(H^d(M; E)).$$

Proposition A.7. *The map*

$$\text{Op}^d(E, E') \rightarrow \mathcal{C}(L^2(M; E), L^2(M; E')), \quad (A, \mathcal{D}) \mapsto A|_{\mathcal{D}}$$

is continuous.

Proof. Denote for brevity $W = H^d(M; E)$, $H = L^2(M; E)$, $H' = L^2(M; E')$. Denote by j the natural embedding $W \hookrightarrow H$. By Proposition A.5, the map $\text{Op}^d(E, E') \subset \mathcal{B}(W, H') \times \text{Gr}(W) \rightarrow \mathcal{C}(W, H')$ is continuous. By definition of $\text{Op}^d(E, E')$, the image of this map is contained in $\mathcal{C}_j(W, H')$. By Proposition A.6, the map $j_*: \mathcal{C}_j(W, H') \rightarrow \mathcal{C}(H, H')$ is continuous. Combining all this, we obtain the continuity of the map $\text{Op}^d(E, E') \rightarrow \mathcal{C}(H, H')$. \square

First order operators and local boundary problems

The main part of the paper deals with first order differential operators and local boundary problems for them. In this subsection we show how the results of the previous subsections can be applied in this case.

Let E be a smooth vector bundle over a smooth compact manifold M with nonempty boundary ∂M . Denote by E_{∂} the restriction of E to ∂M . The trace map $\tau: H^1(M; E) \rightarrow H^{1/2}(\partial M; E_{\partial})$ extends by continuity the restriction map $C^{\infty}(M) \rightarrow C^{\infty}(\partial M)$, $u \mapsto u|_{\partial M}$. For every closed subspace \mathcal{L} of $H^{1/2}(\partial M; E_{\partial})$ its inverse image $\tau^{-1}(\mathcal{L})$ defines a closed subspace of $H^1(M; E)$. Since τ is bounded and surjective, we have the following proposition.

Proposition A.8. *The map*

$$\tau^*: \text{Gr}(H^{1/2}(\partial M; E_{\partial})) \hookrightarrow \text{Gr}(H^1(M; E)), \quad \mathcal{L} \mapsto \tau^{-1}(\mathcal{L}).$$

is continuous.

For $\mathcal{L} \in \text{Gr}(H^{1/2}(\partial M; E_\partial))$ denote by $A_{\mathcal{L}}$ the operator A with the domain

$$\text{dom}(A_{\mathcal{L}}) = \{u \in H^1(M; E) : \tau(u) \in \mathcal{L}\}.$$

The last proposition together with Proposition A.7 implies the gap continuity of the map $(A, \mathcal{L}) \mapsto A_{\mathcal{L}}$ defined on the subspace of $\mathcal{B}(H^1(M; E), L^2(M; E')) \times \text{Gr}(H^{1/2}(\partial M; E_\partial))$ consisting of pairs (A, \mathcal{L}) with closed operator $A_{\mathcal{L}}$.

The following proposition allows us to use this result for *local* boundary problems.

Proposition A.9. *Let N be a smooth compact manifold, E be a smooth Hermitian vector bundle over N . Then the map*

$$(A.4) \quad C^1(N; \text{Gr}(E)) \rightarrow \text{Gr}(H^{1/2}(N; E)), \quad L \mapsto H^{1/2}(N; L) \subset H^{1/2}(N; E)$$

is continuous. Here $\text{Gr}(E)$ denotes the smooth vector bundle over N whose fiber over $x \in N$ is the Grassmanian $\text{Gr}(E_x)$.

Proof. Let P denotes the smooth map from $\text{Gr}(\mathbb{C}^{2n})$ to $\text{End}(\mathbb{C}^{2n})$ which carries $V \in \text{Gr}(\mathbb{C}^{2n})$ into the orthogonal projection P_V of \mathbb{C}^{2n} on V . It induces the continuous map

$$P_* : C^1(N; \text{Gr}(E)) \hookrightarrow C^1(N; \text{End}(E)),$$

which carries a subbundle L of E into the orthogonal projection of E on L . Composing it with the natural continuous inclusion $C^1(N; \text{End}(E)) \hookrightarrow \mathcal{B}(H^{1/2}(N; E))$, we obtain the continuous map

$$Q : C^1(N; \text{Gr}(E)) \hookrightarrow \mathcal{B}(H^{1/2}(N; E)).$$

For each C^1 -subbundle L of E the bounded operator $Q(L)$ is an idempotent with the image $H^{1/2}(N; L)$. Composing Q with the continuous map $\text{Im} : \text{Proj}(H^{1/2}(N; E)) \rightarrow \text{Gr}(H^{1/2}(N; E))$, we obtain the continuity of (A.4). \square

Proof of Proposition 4.1. Denote for brevity $L^2(M; E)$ by H . By Proposition 3.2, the operator A_L is closed for every $(A, L) \in \overline{\text{Ell}}(E)$, so we have the map $\overline{\text{Ell}}^{(0,0)}(E) \hookrightarrow \text{Op}^1(E)$, $(A, L) \mapsto (A, \tau^*(H^{1/2}(L)))$. By Propositions A.8 and A.9, this map is continuous. Thus Proposition A.7 implies the continuity of the map $\overline{\text{Ell}}^{(0,0)}(E) \hookrightarrow \mathcal{C}(H)$, $(A, L) \mapsto A_L$. The image of this map lies in the subspace $\mathcal{RF}^{\text{sa}}(H)$ of $\mathcal{C}(H)$, and the gap topology on $\mathcal{RF}^{\text{sa}}(H)$ coincides with the topology induced from $\mathcal{C}(H)$. This completes the proof of the proposition. \square

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Technion – Israel Institute of Technology, Haifa, Israel
Ural Federal University, Ekaterinburg, Russia
marina.p@tx.technion.ac.il